TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 63 NUMBER 7

CONTENTS

Systems of Distinct Representatives and Linear Programming . . . . . A. J. HOFFMAN AND H. W. KUHN 455
A General Education Course in Pure Mathematics . . . . . HARTLEY ROGERS, JR. 460
Fourier Analysis of Engine Unbalance by Contour Integration . . . . . W. E. BLEICK 466
A Simple Matrix Approach to Linear Differential Equations . . . . . L. C. BARRETT AND C. R. WYLIE, JR. 472
Mathematical Notes . . L. MIRSKY, J. B. CHASSAN, W. FULKS 479
Classroom Notes . . . . . LEONARD CANERS, JOHN R. HATCHER, LUISE LANGE 485
Elementary Problems and Solutions 491
Advanced Problems and Solutions 497
Recent Publications 503
News and Notices 506
The Mathematical Association of America 514
New Sectional Governors of the Association 514
New Members 515
March Meeting of the Michigan Section 517
March Meeting of the Oklahoma Section 520
March Meeting of the Southeastern Section 522
March Meeting of the Southern California Section 531
April Meeting of the Allegheny Mountain Section 532
April Meeting of the Iowa Section 533
April Meeting of the Kentucky Section 536
April Meeting of the Missouri Section 537
April Meeting of the Nebraska Section 539
May Meeting of the Illinois Section 540
May Meeting of the Minnesota Section 542
Calendar of Future Meetings 544

PUBLIC LIBRARY OCT 10 1956 DETROIT

AUGUST-SEPTEMBER 1956

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	E. P. STARKE
HOWARD EVES	L. M. KELLY	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	H. S. ZUCKERMAN
	GEORGE SPRINGER	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to R. W. MACDOWELL, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, W. L. DUREN, JR., University of Virginia
First Vice-President, R. V. CHURCHILL, University of Michigan
Second Vice-President, G. B. PRICE, University of Kansas
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: M. A. BASOCO, H. W. BRINKMANN, P. H. DAUS, H. H. DOWNING, P. D. EDWARDS, N. J. FINE, R. M. FOSTER, PHILIP FRANKLIN, F. F. HELTON, M. S. HENDRICKSON, A. S. HOUSEHOLDER, R. C. HUFFER, C. A. HUTCHINSON, L. W. JOHNSON, G. K. KALISCH, E. C. KIEFER, F. W. KOKOMOOR, R. E. LANGER, Z. L. LOFLIN, L. L. LOWENSTEIN, SAUNDERS MACLANE, E. J. MCSHANE, W. H. MYERS, IVAN NIVEN, MORRIS OSTROFSKY, O. J. RAMLER, J. F. RANDOLPH, C. B. READ, C. R. SHERER, M. F. SMILEY, C. E. SPRINGER, B. M. STEWART, G. B. THOMAS, JR., BERNARD VINOGRADE, M. A. ZORN

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year. To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SYSTEMS OF DISTINCT REPRESENTATIVES AND LINEAR PROGRAMMING*

A. J. HOFFMAN AND H. W. KUHN, National Bureau of Standards and Bryn Mawr College

1. Introduction. In a recent discussion [5] in this MONTHLY of extensions and applications of P. Hall's theorem on distinct representatives of subsets, H. B. Mann and H. J. Ryser presented a sufficient condition that a system of representatives include a prescribed set of elements ([5], Theorems 2.4 and 2.5). The purposes of this note are

(i) to replace the Mann-Ryser condition by a necessary and sufficient condition, and

(ii) to exhibit in the proof two applications of the theory of linear inequalities—specifically, the duality theorem of linear programming [3] and a property of the solutions to transportation problems [4]—to a combinatorial problem.

Although the interested reader can easily concoct a simple, strictly combinatorial proof modeled on the demonstration of Hall's theorem given in [2], it seems worthwhile to call attention to this new method of establishing certain types of combinatorial results. Several theorems that fall into this pattern are treated by various authors in [6].

2. Statement. Let $\mathcal{S}=\{S_1, \cdots, S_n\}$ be a finite collection of subsets of a given set S. A set $R=\{a_1, \cdots, a_n\}$ of n distinct elements of S such that $a_j \in S_j$, for $j=1, \cdots, n$, is called a *system of distinct representatives for* $\mathcal{S}$ (abbreviated SDR). Hall's theorem asserts that the following condition is necessary and sufficient for the existence of an SDR:

(A) For each $q=1, \cdots, n$, the union of any selection of q sets from $\mathcal{S}$ contains at least q distinct elements of S.

Mann and Ryser asked for further restrictions on $\mathcal{S}$ that would insure the existence of an SDR containing l prescribed elements $E=\{e_1, \cdots, e_l\}$—called *marginal elements*. If $t>0$ is the largest number of marginal elements in any S_j, they found that the following condition, combined with (A), suffices for such an SDR:

(B) Every marginal element appears at least t times among the S_j.

To see that this condition is not necessary, consider $S=\{1, 2\}$, $S_1=\{1, 2\}$, $S_2=\{2\}$, $E=\{1, 2\}$. Then $t=2$, (B) is not satisfied, but $R=\{1, 2\}$ is an SDR containing E. Thus (B) is too restrictive; we shall introduce the weaker condition:

(C) For each $p=1, \cdots, l$ and any selection $E'=\{e_{i_1}, \cdots, e_{i_p}\}$ of p marginal elements, the number of sets S_j that meet E' is at least p.

* The preparation of this paper was supported (in part) by the Office of Scientific Research, USAF, and (in part) by the Office of Naval Research.

Condition (C) is clearly necessary: if there is an SDR containing E, by associating to any p marginal elements the sets that they represent, we have immediately the p sets S_j required. It is also obvious that (B) implies (C). Indeed, suppose that some selection E' of p marginal elements meets but p' of the S_j, with $p' < p$. Then the largest possible number of occurrences of marginal elements in these S_j is $p't$. By condition (B), the smallest possible number of such occurrences is pt. Hence $p't \geqq pt$, contradicting $p' < p$.

THEOREM. *Conditions* (A) *and* (C) *are necessary and sufficient for the existence of an SDR containing the prescribed marginal elements* E.

3. Proof of sufficiency. Note that only finite sets S need be considered by restricting S to the finite set of elements needed to verify (A) and (C). List the elements of S as $a_1, \cdots, a_m$ with $a_1 = e_1, \cdots, a_l = e_l$; clearly, $m \geqq n$ by (A) and $n \geqq l$ by (C). It is then convenient to regard the collection $\mathcal{S}$ as given by an m by n incidence matrix:

$$C = (c_{ij}) \quad \text{with} \quad c_{ij} = \begin{cases} 1 \text{ if } a_i \in S_j \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

We define an (l, m, n) *permutation matrix* (for $l \leqq n \leqq m$) to be an m by n matrix of 0's and 1's with

(2) exactly one 1 in each of the first l rows;
(3) no more than one 1 in each of the last $m - l$ rows;
(4) exactly one 1 in each of the n columns.

Observe that each SDR for $\mathcal{S}$ containing E can be identified with an (l, m, n) permutation matrix $P = (p_{ij})$ such that $p_{ij} = 1$ implies $c_{ij} = 1$ for all i and j. Conversely, every (l, m, n) permutation matrix (p_{ij}) supplies an SDR containing E if $p_{ij} = 1$ implies $c_{ij} = 1$.

Next, consider the polyhedral convex set $\mathfrak{X}$ of all m by n matrices $X = (x_{ij})$ such that

$$x_{ij} \geqq 0 \quad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{5}$$

$$\sum_j x_{ij} = 1 \qquad (i = 1, \cdots, l) \tag{6}$$

$$\sum_j x_{ij} \leqq 1 \qquad (i = l + 1, \cdots, m) \tag{7}$$

$$\sum_i x_{ij} = 1 \qquad (j = 1, \cdots, n). \tag{8}$$

The following lemma relates the convex set $\mathfrak{X}$ to the combinatorial problem.

LEMMA. *The linear form* $\sum_{i,j} c_{ij}x_{ij}$ *defined on* $\mathfrak{X}$ *attains its maximum at an* (l, m, n) *permutation matrix.*

Proof. We shall present two proofs of this lemma. The first depends on the fact that a linear form, defined on a compact convex set, achieves its maximum

at an extreme point of that set. (An *extreme point* is any point not expressible as the midpoint of two distinct points of the set.) Clearly, the (l, m, n) permutation matrices, containing only 0's and 1's, are extreme in $\mathfrak{X}$. To show that they are the only extreme points, we need only show that every point in $\mathfrak{X}$ is a convex combination of (l, m, n) permutation matrices. Any $X \in \mathfrak{X}$ can be completed to a square m by m matrix X' by the following rules:

$$x'_{ij} = 0 \qquad (i = 1, \cdots, l \text{ and } j = n+1, \cdots, m) \tag{9}$$

$$x'_{ij} = \frac{1 - \sum_{j=1}^{n} x_{ij}}{m - n} \qquad (i = l+1, \cdots, m \text{ and } j = n+1, \cdots, m) \tag{10}$$

$$x'_{ij} = x_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n). \tag{11}$$

The matrix X' is *doubly-stochastic*: non-negative entries with row and column sums equal to one. As Birkhoff [7] has observed, a theorem* of König [1] (which is, curiously enough, equivalent to Hall's original theorem) implies that X' is the convex combination of *permutation matrices*: matrices composed of 0's and 1's, with exactly one 1 in each row and column. Since X' has only 0's in the first l rows and last $m - n$ columns, the permutation matrices used must also share this property. As a result, the first n columns are (l, m, n) permutation matrices and the desired expression for X has been found.

For a second proof of the lemma, let $\bar{C} = (\bar{c}_{ij})$ be the m by $n+1$ matrix defined by

$$\bar{c}_{ij} = c_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{1'}$$

$$\bar{c}_{i,n+1} = w \qquad (i = 1, \cdots, l), \tag{12}$$

where w is a negative number large in absolute value,

$$\bar{c}_{i,n+1} = 0 \qquad (i = l+1, \cdots, m). \tag{13}$$

Let $\bar{\mathfrak{X}}$ be the set of matrices $\bar{X} = (\bar{x}_{ij})$ satisfying

$$\bar{x}_{ij} \geqq 0 \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n+1) \tag{5'}$$

$$\sum_j \bar{x}_{ij} = 1 \qquad (i = 1, \cdots, m) \tag{6'}$$

$$\sum_i \bar{x}_{ij} = 1 \qquad (j = 1, \cdots, n) \tag{7'}$$

$$\sum_i \bar{x}_{i,n+1} = m - n. \tag{7''}$$

---

* Namely, every term of the expansion of the determinant of an n by n matrix vanishes if and only if the common part of some set of p rows and $n+1-p$ columns contains only zeros.

Consider the problem of maximizing the linear form

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{c}_{ij}\bar{x}_{ij} \tag{14}$$

on the set $\overline{\mathfrak{X}}$. It is clear from (1'), (12), and (13) that the first n columns of the maximizing $\overline{X}$ will be in $\overline{\mathfrak{X}}$ and will maximize $\sum_{i,j} c_{ij}x_{ij}$. However, the problem of maximizing (14) on the set $\overline{\mathfrak{X}}$ is a special case of the so-called "transportation problem" prominent in linear programming, and it is known [4] that there is always a solution, every entry of which is an integer. Therefore, (5'), (6'), and (7') imply that every entry of the solution is either 0 or 1. Hence the first n columns will be an (l, m, n) permutation matrix. This completes the second proof of the lemma.

Now we are in a position to apply the duality theorem of linear programming; a complete discussion of this theorem can be found in [3]. The version needed here relates the following two problems built out of the same data, namely, an M by N matrix $A=(a_{\mu\nu})$, an M-vector $B=(b_\mu)$ and an N-vector $C=(c_\nu)$:

Maximum Problem	*Minimum Problem*	
Maximize $\sum c_\nu\xi_\nu$ for	Minimize $\sum b_\mu\eta_\mu$ for	
$\xi_\nu \geqq 0$	$\sum \eta_\mu a_{\mu\nu} \geqq c_\nu$	$(\nu = 1, \cdots, N)$
$\sum a_{\mu\nu}\xi_\nu \begin{cases} \leqq b_\mu \text{ or} \\ = b_\mu \end{cases}$	$\begin{cases} \eta_\mu \geqq 0 \text{ or} \\ \eta_\mu \text{ unrestricted.} \end{cases}$	$(\mu = 1, \cdots, M)$

The duality theorem then asserts that, if the maximum problem is solvable, then the minimum problem is also solvable and the constrained maximum of $\sum c_\nu\xi_\nu$ equals the constrained minimum of $\sum b_\mu\eta_\mu$.

Resuming the proof of the theorem, the maximum of $\sum_{i,j} c_{ij}x_{ij}$ on $\mathfrak{X}$ (that is, subject to (5)–(8)) is some integer $t \leqq n$. The duality theorem establishes that the minimum of

$$\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \tag{15}$$

for

$$u_i + v_j \geqq c_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{16}$$

and

$$u_i \geqq 0 \qquad (i = l+1, \cdots, m) \tag{17}$$

is also equal to $t \leqq n$.

Let $\bar{u}_1, \cdots, \bar{u}_m, \bar{v}_1, \cdots, \bar{v}_n$ yield a minimum for (15) and have the property that among all solutions it contains the smallest number of non-integral $\bar{u}_i$ and $\bar{v}_j$. We shall show that this implies that the $\bar{u}_i$ and $\bar{v}_j$ are *all* integers. If some are

not integers, let us say that the number of non-integral $\bar{u}_i$ is not less than the number of non-integral $\bar{v}_j$ (if the reverse is true, the argument is changed in an obvious way). Let $\delta = \min(\bar{u}_i - [\bar{u}_i])$ for non-integral $\bar{u}_i$ and define new values by:

$$\bar{u}_i' = \begin{cases} \bar{u}_i & \text{if } \bar{u}_i \text{ is an integer,} \\ \bar{u}_i - \delta & \text{otherwise,} \end{cases} \tag{18}$$

$$\bar{v}_j' = \begin{cases} \bar{v}_j & \text{if } \bar{v}_j \text{ is an integer,} \\ \bar{v}_j + \delta & \text{otherwise.} \end{cases} \tag{19}$$

Clearly $\sum \bar{u}_i' + \sum \bar{v}_j' \leqq \sum \bar{u}_i + \sum \bar{v}_j$. Furthermore, the new values satisfy (16) and (17); the only possibility for violating (16) occurs for a pair of indices (i, j) such that $\bar{v}_j$ is an integer and $\bar{u}_i$ is not. Then

$$\bar{u}_i' + \bar{v}_j' = \bar{u}_i - \delta + \bar{v}_j \geqq [\bar{u}_i] + \bar{v}_j \geqq c_{ij}. \tag{20}$$

Thus, the new values satisfy (16) and (17), do not yield a larger value for (15), but contain fewer non-integral values. This contradiction proves that the chosen $\bar{u}_i$ and $\bar{v}_j$ are all integers.

Next, we shall show that one may take all of the $\bar{u}_i$ and $\bar{v}_j$ to be non-negative. Assume first that some $\bar{v}_j$ is negative. Then, by (16) and (1), it follows that all the $\bar{u}_i$ are positive. Hence, if all of the $\bar{v}_j$ are increased by one and all of the $\bar{u}_i$ are decreased by one, then (16) and (17) will still hold and the value of (15) will not be increased since $n \leqq m$. This process is continued until all of the $\bar{v}_j$ are non-negative.

Assume next that $\min_i \bar{u}_i = -r \leqq -2$. Then, by (16), $\bar{v}_j \geqq r \geqq 2$ for $j = 1, \cdots, n$. If we increase all of the $\bar{u}_i$ equal to $-r$ by one, and decrease all of the $\bar{v}_j$ by one, then (16) and (17) will still hold and the value of (15) will not be increased since $l \leqq n$. Hence any negative $\bar{u}_i$ can be taken equal to -1. Let there be p of these, say $\bar{u}_{i_1} = \cdots = \bar{u}_{i_p} = -1$. Since the $\bar{u}_i$ and $\bar{v}_j$ minimize (15), it follows that a certain number of the $\bar{v}_j$ are equal to 2, and the remainder are 1. Precisely, $\bar{v}_j = 2$ if and only if S_j contains one of the elements $e_{i_1}, \cdots, e_{i_p}$ and there are at least p such by condition (C). If we decrease these $\bar{v}_j$ by one and increase $\bar{u}_{i_1}, \cdots, \bar{u}_{i_p}$ by one, then (16) and (17) will still hold and the value of (15) will not be increased.

Hence we may assume that all of the $\bar{u}_i$ and $\bar{v}_j$ are non-negative integers which, by (16) and (1), are either 0 or 1. Let $\bar{v}_{j_1} = \cdots = \bar{v}_{j_q} = 1$ and the remaining $\bar{v}_j = 0$. Since the $\bar{u}_i$ and $\bar{v}_j$ minimize (15), it follows that $\bar{u}_i = 1$ if and only if a_i is an element of some S_j for which $\bar{v}_j = 0$. By condition (A) there are at least $n - q$ such elements and

$$t = \sum_i \bar{u}_i + \sum_j \bar{v}_j \geqq (n - q) + q = n. \tag{21}$$

As previously remarked, the value t of the maximum program is not greater

than n. Hence it is exactly n and any (l, m, n) permutation matrix that achieves the maximum defines an SDR containing E.

Bibliography

1. Denes König, Theorie der Endlichen and Unendlichen Graphen, Chelsea, New York, 1950.
2. P. R. Halmos and H. E. Vaughan, The marriage problem, Amer. Jour. Math., vol. 72, 1950, pp. 214–215.
3. David Gale, H. W. Kuhn, and A. W. Tucker, Linear programming and the theory of games, Cowles Commission Monograph No. 13, T. C. Koopmans, ed., John Wiley, New York, 1951.
4. G. B. Dantzig, Application of the simplex method to a transportation problem, Cowles Commission Monograph No. 13, T. C. Koopmans, ed., John Wiley, New York, 1951.
5. H. B. Mann and H. J. Ryser, Systems of distinct representatives, this MONTHLY, vol. 60, 1953, pp. 397–401.
6. Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Annals of Math. Study No. 38, in press.
7. G. Birkhoff, Two remarks on linear algebra, Rev. Univ. Nac. Tucuman, Series A, vol. 5, 1946, pp. 147–148.

A GENERAL EDUCATION COURSE IN PURE MATHEMATICS

HARTLEY ROGERS, JR., Massachusetts Institute of Technology

A description follows ot a course given at Harvard in the spring semester of 1955; the course was sponsored jointly by the Mathematics Department and the Committee on General Education as an elective course not normally open to freshmen. The course appeared to be quite successful. There has been considerable interest in such courses in recent years and although none of the material presented was new to such efforts, there seemed to be enough that was distinctive in arrangement and emphasis to warrant a brief report.*

Enrollment was limited generally, though not inflexibly, to students with an honors average. The only prerequisite was a high school mathematics background that included synthetic geometry. Students with college analysis courses were also admitted, but those with courses in modern algebra or projective geometry were not. The course had 35 regular students and about 15 auditors. About half had had no college mathematics. More than half were concentrators in Humanities or Social Sciences. About half were seniors, the rest juniors and sophomores.

The course was meant to explain, within obvious limitations, something of

* The course was first offered in 1954. It is in a continuing state of development in both form and content. It was given in 1955 by the author and in the preceding year by Professor A. M. Gleason, to whom the author is indebted for valuable suggestions.

the spirit and method of modern pure mathematics. In the arrangement of material for the course, a number of basic preliminary problems were faced. The first was that of deciding how much of the course should be presentation of axiomatic method as method and how much should be communication of substantive mathematical subject matter. It was felt that method was more important for the course, but it was also recognized that standards of significance and beauty in mathematics are largely determined by the extant body of subject matter. The problem was resolved by largely devoting lectures to the axiomatic method, while adopting Courant and Robbins' *What is Mathematics?* as the principal reading text. Each student also wrote a term paper reporting on some substantive topic. A second problem was that of choosing between a qualitative survey and a careful, close exposition of material. This was resolved by varying the lecture treatment from topic to topic; some topics were presented in careful detail, others in looser historical summary. A third problem was that of arranging the course so that the student could discover good reasons for being interested in pure mathematics. Two approaches seemed possible: emphasis could be allowed to rest on the intrinsic intellectual and aesthetic interest of abstract mathematics; or an attempt could be made to show the student how the tools of modern applied mathematics are not naively visualizable, and require an understanding of rigor and the axiomatic method. While these approaches were not incompatible, it seemed wisest, in view of purposes and limited time, to choose topics with the first of these approaches in the foreground. The existence of these three problems and the overall limitations of the course were emphasized to the students from the start. It was pointed out that they could examine only a very small part of an active and changing discipline, and they were urged to remain flexible and uncommitted in their initial attitudes toward it.

The course occupied 37 one hour lectures. There was a mid-term hour test and a three hour final examination. There were weekly homework exercises and occasional more difficult extra-credit problems for students of more advanced background or ability. At intervals through the term, some of the better logical "brain-teasers" were also given as unofficial problems. A term paper was required in the form of a report on outside reading of the student's choice. The student conferred with the teacher about this choice, and permissible topics depended on the student's background. Several students with no analysis undertook to learn elementary calculus from the later chapters of Courant and Robbins and did so with considerable success. Others ranged in their subjects from the real number system to the Theory of Games and elementary theorems in Artin's work on braids.

Courant and Robbins was adopted as the principal text. While parts of it are rather sophisticated for a course at this level, and while the net official reading assignment was consequently small, it was nevertheless felt to be admirably suited to student browsing. A list of suggested books for supplementary reading was given out at the beginning of the term. (This is given below.) It was thought

important to give the student definite warning and guidance in this regard as there is enormous variation of quality and purpose among the usual library volumes in the introductory and "semi-popular" areas.

Before a brief outline of the lecture topics is given, several omissions deserve special mention. No logical symbolism, as such, was presented. Matters of normative linguistic convention quickly arose anyway and were dealt with as such. It was felt that this was more efficient in the time available and that a separate treatment of logic might be more distraction than help in achieving the purposes of the lectures. Much can, of course, be said against this view. Conventions such as vacuous implication, the empty set, and existential import, had to be made clear, and the logical quantifiers and "if . . . then . . . " arrows were occasionally used as abbreviations; nevertheless natural language remained the basic logical medium throughout the term. No rigorous treatment was given of the real number system. Examples from algebra and geometry seemed better suited to introductory illustrations of rigorous argument. Furthermore, in an initial exposure to the axiomatic method, a student's partial knowledge of the real numbers may somewhat obscure for him the necessity and productivity of the axiomatic approach. No topics from analysis were given. This was partly because of varied background among the students and partly because of the relatively greater logical complexity of elementary notions of analysis. The central position of analysis in the body of mathematics was strongly emphasized to the students, and supplementary reading was recommended. The omitted areas of logic and analysis were urged as possible sources of material for term papers, and a number of students chose to work in them.

An outline of the lecture subject matter follows.

I. The first two lectures were devoted to presenting, slowly and carefully, proofs of the irrationality of $\sqrt{2}$, the infinitude of the primes, and the unique prime factorization of the integers. The proofs were based on "known" properties of the integers rather than on explicitly listed axiomatic assumptions. The theorems served two purposes: 1) they showed the students the degree of logical facility that would be expected of them during the term (as was intended, this prompted several to transfer out of the course), and 2) the theorems gave vivid examples of the productivity of rigorous proof, even in dealing with familiar and supposedly well understood objects. This was most valuable, as many of the students seemed to have come with the feeling that mathematics is a matter of ingenious definition followed by relatively straightforward application and that proof is a kind of scholastic exercise whereby one "justifies" what one already sees to be true. The theorems also served as an illustration of the subtleties of indirect proof. The proof of unique factorization was the elegant and seldom used proof that appears in Courant and Robbins. No preliminary divisibility lemma is necessary, and mathematical induction is disguised and compressed into an initial choice of the least integer not possessing unique factorization.

II. One lecture was then given over to a qualitative preview discussion of the axiomatic method. It seemed worthwhile to expose the students to such a

preliminary summary, even though, as was announced to them, many were expected not to follow the lecture entirely.

III. Several lectures were then spent on a critique of the proofs of Euclidean geometry as based on Euclid's original axioms. Emphasis was placed on the grosser tacit assumptions rather than on the finer logical inadequacies. The more celebrated gaps were pointed out and some of the traditional "false" proofs ("every triangle is isosceles") were also given. (We remark that a good example of a tacit assumption occurs in Euclid I, 1, the construction of an equilateral triangle on a given line segment. Here Euclid assumes that the arcs drawn from each end of the segment will intersect.)

IV. Axioms for synthetic finite affine geometry were then presented as the first major topic of the term. This material is well known as a highly successful pedagogical illustration of axiomatic development. It goes back to Young, *Lectures on Fundamental Concepts of Algebra and Geometry* (1911) and has appeared in a number of more recent texts. It well exhibits the relation between a theory and its specific models and the possibility—new to many of the students—of having a non-categorical theory. Once the existence of the various non-intuitive models is understood, the necessity for a careful axiomatic process becomes clear. The lectures presented the material slowly, first using nonsense words for "point" and "line" and then using those words themselves with the observation that they were to function like the nonsense words. A particular virtue of this material is that theorems of considerable generality can soon be reached—from a proof that every two lines have the same number of points (where the notion of one-one correspondence must be used) to a sequence of non-trivial structure theorems (every finite model has n^2 points, n^2+n lines, *etc.*). The structure theorems were presented as such to the student and it was pointed out that they provided a good example in microcosm of much modern mathematical activity. In addition, a number of unsolved problems could be stated (existence of a 10 point line, *etc.*), and it could be remarked that there was current research going on in the area. The development was also used to provide clear illustrations of the process of definition, the use of counter-examples, and the notions of consistency, independence and incompleteness of an axiom system, (the use of finite geometry in this respect is well described in Wilder, *Introduction to the Foundations of Mathematics*).

The material also furnished a good opportunity for discussion of such matters as: the motives that may exist for examining a particular system; the unexpected fruitfulness and generality that may appear; and the overall flexibility and control inherent in the axiomatic approach. Axioms for finite projective geometry were also presented and the corresponding structure theory derived. Final theorems relating this to the affine theory were proved. The subject of finite geometry occupied seven lectures. It was treated, for the most part, in full detail.

V. Four lectures were devoted to the Hilbert axiomatization of Euclidean geometry, with the purpose of conveying an understanding of the axioms them-

selves and then giving a brief indication of how the usual theorems grow out of them. One lecture was given on abstract relations (preparation for "between" and "congruent") and then the axioms were presented in detail along with proofs of a few of the earlier and simpler theorems. The presentation in the 1930 German edition of Hilbert, *Grundlagen der Geometrie*, was used as a source. This is somewhat better than in the 1899 edition, which appears in English translation. Material from Hilbert together with additional comment was distributed to the students in mimeograph form.

VI. Six lectures were used to give an outline of the origins and discovery of non-Euclidean geometry. Material was presented in roughly historical sequence. The mathematical treatment was qualitative to the extent that synthetic proofs at about the level of school geometry were given. Wolfe, *Non-Euclidean Geometry*, proved to be a good source and reference. Attention was directed first to propositions equivalent to the fifth postulate and then to the use of tacit assumptions or bad logic in various historical false proofs of the postulate. The elements of non-Euclidean geometry itself were then described and the proof of consistency with Euclidean geometry outlined. Two matters that were naturally emphasized were: 1) the importance of keeping theory and model distinct; and 2) the fact that much excellent mathematics can historically precede full axiomatization, since adequate conventional standards can tacitly appear among good mathematicians, (*e.g.*, by 1800 the false proofs of the fifth postulate were recognized as such).

VII. Seven lectures then gave a careful treatment of elementary material from algebra. Axioms for a commutative ring were given and various examples of rings supplied. A few of the simplest theorems were proved. Specializations to integral domain, field and Boolean ring were briefly investigated, and the notion of group was described.

VIII. The subject of transfinite cardinal numbers was considered next. While not thought to have the basic relevance of some of the other material in the course, it furnished good logical exercise, had a special metaphysical interest for many of the students and paved the way for a final discussion of foundational matters. Treatment was non-axiomatic and proceeded by a combination of illustration, logical argument and statement of results. Relevant properties of the finite cardinals were first examined and then modified appropriately to lead to the transfinite case. The relation of cardinal equivalence was made the central concept for as long as possible, and then the transfinite numbers themselves were introduced. Cantor's theorem was proved, both in decimal-diagonal and in its general form. Examples were given of various enumerable and non-enumerable sets. Addition and multiplication were defined, and the algebraic properties of the arithmetic of cardinals were discussed. The subject occupied five lectures.

IX. The subject of cardinal numbers had emphasized the central importance of the concept of *collection*. The students had also seen its basic role in their study of geometry. It was thus natural and easy to raise the question of what

constituted legitimate argument about collections. Furthermore, the counter-intuitive quality of the results of the transfinite theory had aroused student interest in finding an axiom system for that theory. The final three lectures of the term were therefore devoted to a qualitative discussion of foundations and of axiomatic set theory. The earlier proof of Cantor's theorem made possible a statement of Cantor's "greatest cardinal" paradox. Russell's paradox was stated as well, and the general history of the foundational crisis was described. The nature and purposes of logical formalization were outlined, and various formal means for avoiding the paradoxes were briefly indicated. The role of an axiomatic set theory as a common standard of argument rather than common means of expression was pointed out. It was observed (as had been done several times earlier in the term) that all of usual mathematics can be put into the symbolism of such a theory. The existence of a variety of such theories was mentioned, together with Gödel's results on incompleteness and on impossibility of a consistency proof. While these last lectures were highly qualitative and necessarily inadequate, they seemed to give the students a fairly satisfactory understanding of the main issues considered.

The course omitted much standard material, but it was felt at the conclusion that some feeling had been communicated for the blending of rigor and intuition that makes up contemporary pure mathematics, and that interested students would have a fair, if rudimentary, idea of what to expect in further exploration of the subject.

Reference List

* What is Mathematics? Courant and Robbins
Lectures on Fundamental Concepts of Algebra and Geometry, J. W. Young
Introduction to Foundations of Mathematics, R. L. Wilder
Principles of Mathematics, Allendoerfer and Oakley
The Anatomy of Mathematics, Kershner and Wilcox
Introduction to Mathematics, Cooley, Gans, Kline, and Wahlert
Men of Mathematics, E. T. Bell
Mathematics and the Imagination, Kasner and Newman
Non-Euclidean Geometry, H. E. Wolfe
Geometry and the Imagination, Hilbert and Cohn-Vossen
Foundations of Geometry, D. Hilbert
Foundations of Analysis, E. Landau
Survey of Modern Algebra, Birkhoff and MacLane
Differential and Integral Calculus, R. Courant
Mathematical Snapshots, H. Steinhaus
Fundamentals of Mathematics, M. Richardson
Concepts and Structure of Mathematics, Univ. of Chicago College Math. Staff

* Required.

FOURIER ANALYSIS OF ENGINE UNBALANCE BY CONTOUR INTEGRATION*

W. E. BLEICK, U. S. Naval Postgraduate School

1. Introduction. A Fourier analysis of the dynamic unbalance of a reciprocating engine by contour integration may be of pedagogic interest in teaching the engineering applications of complex variable theory. Figure 1 gives the notation for the displacement variables s, θ and ϕ, which are the piston displacement, the crank angle, and the connecting rod angle, respectively. The ratio of the crank length r to the connecting rod length l is denoted by e. The complete analysis of dynamic unbalance follows directly from the Fourier series for s, $\dot{\phi}$, $\dot{\phi}^2$ and $\dot{s}^2$. Each of these is treated here in turn.

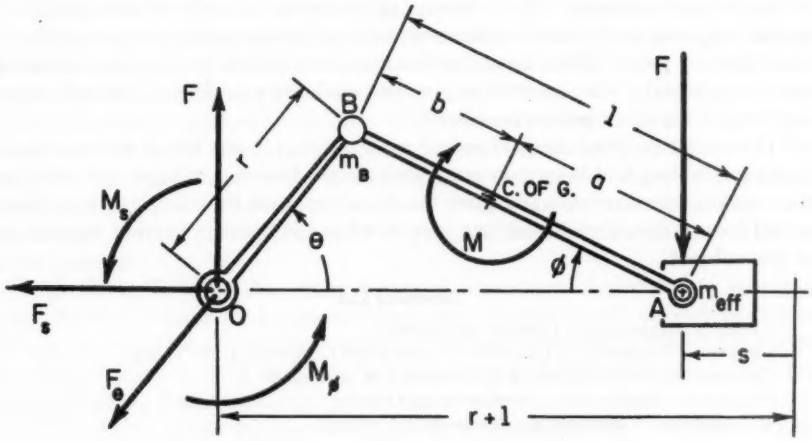

FIG. 1

2. Fourier Series for s. The piston displacement s is expressed in terms of the crank angle θ by

$$s = r[1 - \cos\theta + 1/e - \sqrt{1 - e^2\sin^2\theta}/e]. \tag{1}$$

The Fourier series for s is

$$s = r\left[a_0 - \cos\theta + \sum_{n=1}^{\infty} a_{2n}\cos 2n\theta\right]. \tag{2}$$

The value of a_0, the average value of s/r, is found by integration to be

$$a_0 = 1 + 1/e - 2E(e)/\pi e, \tag{3}$$

* Presented to the Northern California Section of the Mathematical Association of America, January 15, 1955.

where $E(e)$ is the complete elliptic integral of the second kind and modulus e. The coefficient a_{2n} for $n>0$ is

$$a_{2n} = -\int_0^{2\pi} \sqrt{1 - e^2 \sin^2\theta}(\cos\theta + j\sin\theta)^{2n} d\theta/\pi e, \tag{4}$$

using De Moivre's theorem for $\cos 2n\theta + j \sin 2n\theta$, and noting that the imaginary part of the integral vanishes since the radical is an even function of θ. Reference to [1] shows that the integral for a_{2n} may be converted to a contour integral by the imaginary exponential substitution

$$z = e^{j\theta} = \cos\theta + j\sin\theta. \tag{5}$$

The result is

$$a_{2n} = j\int_{ABC} z^{2n-2}\sqrt{(z^2 + \alpha^2)(z^2 + \alpha^{-2})}\, dz/2\pi, \tag{6}$$

where

$$\alpha = (1 - \sqrt{1 - e^2})/e, \tag{7}$$

and the path of integration is counterclockwise around the unit circle $|z| = 1$ in the complex plane, *i.e.*, the path ABC in Figure 2. In order that the integrand in (6) may be analytic in a suitable region, introduce branch cuts, indicated by heavy lines in Figure 2, along the imaginary axis from $-j\infty$ to $-j/\alpha$, from $-j\alpha$ to $+j\alpha$, and from $+j/\alpha$ to $+j\infty$, and consider the branch of the radical which is defined to have the value $\sqrt{r_1 r_2 r_3 r_4}\exp[j(\phi_1+\phi_2+\phi_3+\phi_4)/2]$, where $-\pi/2 \leqq \phi_1 < 3\pi/2$, $-\pi/2 \leqq \phi_2 < 3\pi/2$, $3\pi/2 \leqq \phi_3 < 7\pi/2$ and $\pi/2 \leqq \phi_4 < 5\pi/2$. The symbols in this definition are given in Figure 2.

By use of the branch thus defined, the path of integration may be deformed to the path ADC along the two sides of the imaginary axis and around the branch point $+j\alpha$, as shown in Figure 2. After making the substitution $z = jy$, the expression for a_{2n} becomes

$$a_{2n} = 2(-1)^n \int_0^{\alpha} y^{2n-2}\sqrt{(\alpha^2 - y^2)(\alpha^{-2} - y^2)}\, dy/\pi. \tag{8}$$

The further substitution $y = \alpha u$ gives

$$a_{2n} = 2(-1)^n \alpha^{2n-1}[\alpha^4 Y_{2n+2} - (1 + \alpha^4) Y_{2n} + Y_{2n-2}]/\pi, \tag{9}$$

where Y_{2n} is the elliptic integral

$$Y_{2n} = \int_0^1 u^{2n} du/\sqrt{(1 - u^2)(1 - \alpha^4 u^2)}. \tag{10}$$

The usual three-term recurrence formula for Y_{2n} is obtained from the differentiation

$$(d/du)u^{2n-1}\sqrt{(1-u^2)(1-\alpha^4u^2)}$$

$$(11)\qquad = \frac{(2n+1)\alpha^4u^{2n+2} - 2n(1+\alpha^4)u^{2n} + (2n-1)u^{2n-2}}{\sqrt{(1-u^2)(1-\alpha^4u^2)}}.$$

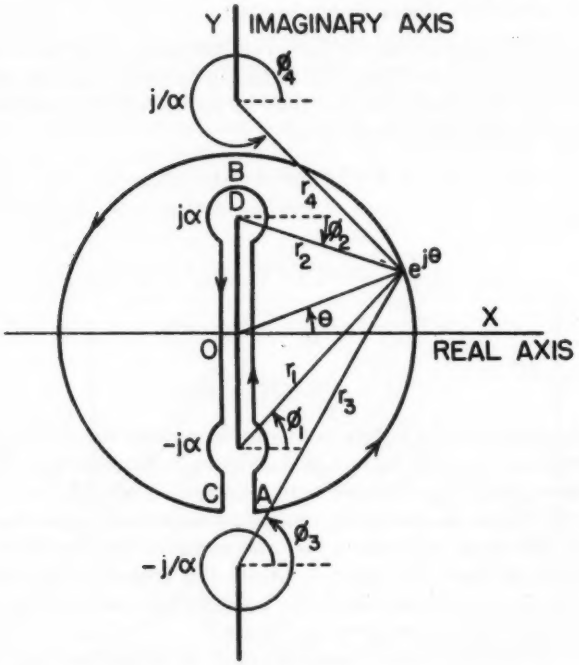

FIG. 2

Integration of (11) from $u=0$ to $u=1$ furnishes the recurrence formula

$$(12)\qquad (2n+1)\alpha^4Y_{2n+2} - 2n(1+\alpha^4)Y_{2n} + (2n-1)Y_{2n-2} = 0.$$

The initial values required in the use of (12) are

$$Y_0 = K(\alpha^2) \quad \text{and} \quad Y_2 = [K(\alpha^2) - E(\alpha^2)]/\alpha^4,$$

where $K(\alpha^2)$ and $E(\alpha^2)$ are the complete elliptic integrals of the first and second kinds, respectively, for modulus α^2. Useful tables of Y_0 and Y_2 are given in [2] on pages 85 and 83, respectively. Use of (9) and (12) makes possible the computation of a_{2n} for all values of n. Thus, for $n=1$,

$$(13)\qquad a_2 = -2[(1+\alpha^4)E(\alpha^2) - (1-\alpha^4)K(\alpha^2)]/3\alpha^2\pi.$$

Lengthy formulae could be given here for other a_{2n} coefficients, but they are calculated more conveniently by the numerical use of (9) and (12).

3. Fourier series for $\dot{\phi}$. The value of $\dot{\phi}$ is $\dot{\phi}=e\dot{\theta}\cos\theta/\sqrt{1-e^2\sin^2\theta}$, and its Fourier series is

$$\dot{\phi} = \dot{\theta}\sum_{n=0}^{\infty} b_{2n+1}\cos(2n+1)\theta. \tag{14}$$

The imaginary exponential substitution of (5) and the deformed contour ADC of Figure 2 may be used to give

$$b_{2n+1} = \int_{ADC} z^{2n}(z^2+1)dz/\pi j\sqrt{(z^2+\alpha^2)(z^2+\alpha^{-2})}. \tag{15}$$

The integrals on the infinitesimal circles surrounding the poles $z=\pm j\alpha$ may be shown to vanish. The successive substitutions $z=jy$ and $y=\alpha u$ then give

$$\begin{aligned} b_{2n+1} &= 4(-1)^n\alpha^{2n+1}\int_0^1 (u^{2n}-\alpha^2u^{2n+2})du/\pi\sqrt{(1-u^2)(1-\alpha^4u^2)} \\ &= 4(-1)^n\alpha^{2n+1}(Y_{2n}-\alpha^2Y_{2n+2})/\pi. \end{aligned} \tag{16}$$

Thus, for $n=0$,

$$b_1 = 4[E(\alpha^2)-(1-\alpha^2)K(\alpha^2)]/\pi, \tag{17}$$

and for $n=1$,

$$b_3 = 4[(\alpha^4-3\alpha^2+2)K(\alpha^2)-(2\alpha^4-3\alpha^2+2)E(\alpha^2)]/3\pi\alpha^2. \tag{18}$$

4. Fourier series for $\dot{\phi}^2$. The Fourier series for $\dot{\phi}^2=e^2\dot{\theta}^2\cos^2\theta/(1-e^2\sin^2\theta)$ is

$$\dot{\phi}^2 = \dot{\theta}^2\left[c_0/2+\sum_{n=1}^{\infty} c_{2n}\cos 2n\theta\right]. \tag{19}$$

The imaginary exponential substitution of (5) and a path of integration around the unit circle $|z|=1$ gives

$$c_{2n} = \int_{ABC} z^{2n-1}(z^2+1)^2dz/\pi j(z^2+\alpha^2)(z^2+\alpha^{-2}). \tag{20}$$

For $n=0$ there are simple poles within the contour at the origin and $\pm j\alpha$. Calculation of the residue sum gives

$$c_0 = (1+\alpha^2)e^2. \tag{21}$$

For $n>0$ there are simple poles at $\pm j\alpha$ only, and calculation of the residue sum gives

$$c_{2n} = 2(-1)^{n-1}\alpha^{2n}\sqrt{1-e^2}. \tag{22}$$

5. Fourier series for $\dot{s}^2$. Since $\dot{s}=r(\dot{\theta}+\dot{\phi})\sin\theta$, it follows that

$$\dot{s}^2 = r^2(1-\cos 2\theta)(\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2)/2. \tag{23}$$

Substituting in (23) for ϕ and ϕ^2 from (14) and (19), respectively, and using the addition theorem for cosines, gives

$$s^2 = (r^2\dot{\theta}^2/2) \sum_{k=0}^{\infty} d_k \cos k\theta, \tag{24}$$

where

$$\begin{aligned} d_0 &= 1 + (c_0 - c_2)/2, \\ d_1 &= b_1 - b_3, \\ d_2 &= c_2 - 1 - (c_0 + c_4)/2, \\ d_{2n+1} &= 2b_{2n+1} - b_{2n-1} - b_{2n+3} \text{ for } n > 0, \end{aligned} \tag{25}$$

and

$$d_{2n} = c_{2n} - (c_{2n-2} + c_{2n+2})/2 \text{ for } n > 1.$$

Substitution from (16), (21) and (22) into (25) gives

$$\begin{aligned} d_0 &= 1 + \alpha^2, \\ d_1 &= -4a_2 > 0, \\ d_2 &= -1 - \alpha^4, \\ d_{2n+1} &= 4(-1)^n\alpha^{2n-1}(Y_{2n-2} + \alpha^2 Y_{2n} - \alpha^4 Y_{2n+2} - \alpha^6 Y_{2n+4})/\pi \text{ for } n > 0, \\ &= 2a_{2n} + 2a_{2n+2} + 4b_{2n+1}/e^2, \end{aligned} \tag{26}$$

and

$$d_{2n} = (-1)^{n-1}\alpha^{2n-2}(1 - \alpha^4) \text{ for } n > 1.$$

Note, that, for customary values of e, d_2 is greater in magnitude than d_1 and of opposite sign.

6. Dynamic unbalance components. In the analysis of dynamic unbalance it is customary, *e.g.* [3], because of balancing considerations, to replace the connecting rod by a statically and dynamically equivalent connecting rod having its mass concentrated at the two ends of the rod. To preserve static and translational dynamic equivalence, all that is necessary is to maintain the center of gravity invariant. To this end the portions of the mass m of the connecting rod concentrated at the crank pin B and wrist pin A of Figure 1 are $m_B = ma/l$ and $m_A = mb/l$, respectively, where the distances of wrist pin and crank pin from the center of gravity are a and b, respectively. This substitution may also be regarded as yielding an equivalent engine having a massless connecting rod with a piston of effective mass m_{eff} greater than its true mass by the amount $m_A = mb/l$, and a crank of effective mass greater than its true mass by the amount $m_B = ma/l$. To achieve angular dynamic equivalence, it is necessary to apply to the connecting rod an auxiliary external torque $M = (mab - \bar{I})\ddot{\phi}$ where $\bar{I}$ is the moment of inertia of the rod about its center of gravity. This additional

torque is required in order to accelerate the increase in the moment of inertia caused by concentrating the mass at the ends of the connecting rod. A free body of the equivalent engine is given in Figure 1, the symbols for the various forces and torques being explained in the following paragraphs. The free body omits static and frictional forces. To simplify the expressions for dynamic unbalance given below, the customary assumption of constant crank angular velocity $\dot{\theta}$ is made, although this assumption is not necessary in the present method of analysis.

The Fourier components of dynamic unbalance may now be obtained in a very direct fashion by time differentiation of the Fourier kinematic series previously obtained. The shaking force F_s on the crank shaft due to the motion of the effective piston mass m_{eff} is found by differentiating (2) to be

$$\begin{aligned} F_s &= m_{eff}\ddot{s} \\ &= m_{eff}r\dot{\theta}^2\left(\cos\theta - \sum_{n=1}^{\infty} 4n^2a_{2n}\cos 2n\theta\right). \end{aligned} \tag{27}$$

The torque M_s about the crank shaft, due to the motion of the effective piston mass, is found from the work and energy relationship

$$M_s\dot{\theta}dt = (d/dt)(m_{eff}\dot{s}^2/2)dt \tag{28}$$

and (24) to be

$$M_s = -(m_{eff}r^2\dot{\theta}^2/4)\sum_{k=1}^{\infty} kd_k \sin k\theta. \tag{29}$$

The torque M_ϕ about the crank shaft, due to the auxiliary torque M acting on the equivalent connecting rod, is found from the virtual displacement relationship

$$\begin{aligned} M_\phi\dot{\theta}dt &= -M\dot{\phi}dt \\ &= -(d/dt)[(mab - \bar{I})\dot{\phi}^2/2]dt, \end{aligned} \tag{30}$$

and (19) to be

$$M_\phi = (mab - \bar{I})\sum_{n=1}^{\infty} nc_{2n}\sin 2n\theta. \tag{31}$$

The forces F, perpendicular to the cylinder center line, may be obtained by noting that, for constant crank angular velocity, the moment of momentum of the equivalent engine about the crank shaft O of Figure 1 is constant. Hence the resultant torque about O must vanish, giving

$$M_s + M_\phi - M - (r + l - s)F = 0. \tag{32}$$

It is unnecessary to discuss the rotating force F_θ, 180° out of phase with the crank angle θ, due to the centripetal acceleration of the effective crank mass,

since a Fourier analysis is not involved.

It is beyond the scope of the present paper to discuss dynamic unbalance in multicylinder engines, where the net effect of certain Fourier components may be eliminated.

References

1. Churchill, R. V., Introduction to Complex Variables and Applications, McGraw-Hill Book Co., New York, 1948, p. 130.

2. Jahnke, E. and Emde, F., Tables of Functions with Formulae and Curves, Dover Publications, New York, 1945.

3. Root, R. E., Dynamics of Engine and Shaft, John Wiley and Sons, Inc., New York, 1932, p. 7.

A SIMPLE MATRIX APPROACH TO LINEAR DIFFERENTIAL EQUATIONS

L. C. BARRETT AND C. R. WYLIE, JR., University of Utah

In texts on elementary differential equations the subject of systems of linear differential equations with constant coefficients is frequently dismissed with little more than an example or two. From time to time suggestions have been made for improving the presentation through the use of ideas from the elementary theory of matrices,† but to date these seem to have had little effect on the books available for use in the classroom. This is perhaps because such discussions have been based on the canonical form

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \qquad i = 1, 2, \cdots, n,$$

which can be obtained from a system of differential equations as they actually arise only through a considerable amount of preliminary algebra. It is the purpose of this note to suggest a more practical matrix approach to the solution of a system of equations in their original form

$$\begin{aligned} a_{11}(D)x_1 + \cdots + a_{1n}(D)x_n &= f_1(t) \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{n1}(D)x_1 + \cdots + a_{nn}(D)x_n &= f_n(t), \end{aligned} \tag{1}$$

where the a_{ij} are polynomials in $D(\equiv d/dt)$ with constant coefficients. In our

† C. M. Cramlet, Linear differential equations with constant coefficients, this MONTHLY, vol. 45, 1938, p. 162.

J. S. Frame, On the explicit solution of simultaneous linear differential equations with constant coefficients, this MONTHLY, vol. 47, 1940, p. 35.

See also Richard Bellman, Stability Theory of Differential Equations, p. 12, McGraw-Hill Book Co., New York, 1953.

work we shall actually use little more than the matrix notation; specifically, we need only a knowledge of how to add, multiply, and differentiate matrices.

Introducing the matrices

$$A(D) = \begin{Vmatrix} a_{11}(D) & \cdots & a_{1n}(D) \\ \cdot & \cdots & \cdot \\ a_{n1}(D) & \cdots & a_{nn}(D) \end{Vmatrix}, \quad X = \begin{Vmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{Vmatrix}, \quad F = \begin{Vmatrix} f_1(t) \\ \cdot \\ \cdot \\ f_n(t) \end{Vmatrix},$$

(1) can be written in the form

$$AX = F. \tag{2}$$

By analogy with a single differential equation we are led to consider the related homogeneous equation

$$AX = 0 \tag{3}$$

and to attempt to find solutions of it of the form

$$X = Ce^{mt} \tag{4}$$

where

$$C = \begin{Vmatrix} c_1 \\ \cdot \\ \cdot \\ c_n \end{Vmatrix}$$

is a constant, non-trivial coefficient matrix. Substitution of (4) in (3) yields the equation

$$A(m)Ce^{mt} = 0$$

which requires that

$$A(m)C = 0 \tag{5}$$

or finally

$$|A(m)| = 0. \tag{6}$$

This, of course, is the usual determinantal characteristic equation. It is a polynomial equation in m of degree k, say, each of whose roots, m_j, gives rise to a solution of (3) of the form (4) provided C is suitably determined. For each m_j this can be done by applying familiar algebraic procedures to (5). The resulting C_j is of course not a unique matrix but a matrix determined only to within an arbitrary scalar multiplier b_j. Since linear combinations of particular solutions are also solutions,

$$X = \sum_j b_j C_j e^{m_j t} \equiv \sum_j b_j X_j$$

is a solution of (3) for all values of the arbitrary scalar multipliers b_j. If the roots

of (6) are all distinct this is the complete solution of (3) and the complementary function of (2).

As in the case of a single differential equation, if the set $\{m_j\}$ includes one or more pairs of conjugate complex roots it is desirable to reduce the corresponding complex exponential solutions to a purely real form. To see how this can be accomplished let $m_1, m_2 = p \pm iq$ be a pair of conjugate complex roots of (6) and let C_1 be a particular coefficient matrix corresponding to $m_1 = p + iq$. Then by taking conjugates throughout (5) it is clear that $C_2 = \overline{C}_1$ is a coefficient matrix corresponding to $m_2 = p - iq$. Hence we have the two particular solutions

$$C_1 e^{(p+iq)t} \quad \text{and} \quad \overline{C}_1 e^{(p-iq)t}.$$

By combining these as follows and applying the Euler formulas we obtain two independent real particular solutions:

$$\begin{aligned} \tfrac{1}{2}[C_1 e^{(p+iq)t} + \overline{C}_1 e^{(p-iq)t}] &= e^{pt}[\tfrac{1}{2}(C_1 + \overline{C}_1)\cos qt + \tfrac{1}{2}i(C_1 - \overline{C}_1)\sin qt] \\ &= e^{pt}[\mathfrak{R}(C_1)\cos qt - \mathfrak{I}(C_1)\sin qt], \\ -\tfrac{1}{2}i[C_1 e^{(p+iq)t} - \overline{C}_1 e^{(p-iq)t}] &= e^{pt}[-\tfrac{1}{2}i(C_1 - \overline{C}_1)\cos qt + \tfrac{1}{2}(C_1 + \overline{C}_1)\sin qt] \\ &= e^{pt}[\mathfrak{I}(C_1)\cos qt + \mathfrak{R}(C_1)\sin qt]. \end{aligned} \tag{7}$$

In many cases this method of determining the necessary relations among the coefficients of solutions of (3) of the form

$$x_j = a_j e^{pt}\cos qt + b_j e^{pt}\sin qt$$

is simpler than the usual process of substituting these expressions into the original differential equations, collecting terms, and equating the resulting coefficients to zero.

If $|A(m)| = 0$ has a double root, say $m = m_1$, we proceed very much as in the case of a single differential equation. Of course

$$X_1 = C_1 e^{m_1 t}$$

is one solution of (3). However as a second independent solution we must try not $C_2 t e^{m_1 t}$, as strict analogy with the solution-process for a single differential equation would dictate, but rather†

$$X_2 = C_2 e^{m_1 t} + C_3 t e^{m_1 t}.$$

Substituting this into (3), observing that for any polynomial operator $P(D)$

$$P(D)te^{mt} = P(m)te^{mt} + P'(m)e^{mt},$$

and recalling the definition of the derivative of a matrix, we obtain

$$A(m_1)C_2 e^{m_1 t} + A(m_1)C_3 t e^{m_1 t} + A'(m_1)C_3 e^{m_1 t} = 0.$$

This requires that

† The term $C_2 e^{m_1 t}$ must be retained because in general C_2 will not be a scalar multiple of C_1 and hence in constructing the general solution the term $C_2 e^{m_1 t}$ cannot be absorbed in the term $C_1 e^{m_1 t}$ as is necessarily the case for a single differential equation.

$$A(m_1)C_3 = 0 \quad \text{and} \quad A(m_1)C_2 = -A'(m_1)C_3. \tag{8}$$

The first of these equations shows that to within an arbitrary scalar factor C_3 is the same as C_1. The second constitutes a system of nonhomogeneous linear algebraic equations from which the elements of C_2 can be found by familiar processes. Its solution is of the form

$$C_2 = \lambda C_1 + {}^*C_2$$

where ${}^*C_2$ is a "particular integral" of the (algebraic) system. Hence, omitting the term $\lambda C_1 e^{m_1 t}$ from X_2 since it can obviously be absorbed in X_1, and taking $C_3 = C_1$, we obtain

$$X_2 = {}^*C_2 e^{m_1 t} + C_1 t e^{m_1 t}$$

as a second independent particular solution of (3) corresponding to the double root m_1 of (6).

Roots of the characteristic equation of higher multiplicity are handled in essentially the same way. For instance if $|A(m)| = 0$ has a triple root $m = m_1$, then not only are

$$C_1 e^{m_1 t} \quad \text{and} \quad {}^*C_2 e^{m_1 t} + C_1 t e^{m_1 t}$$

solutions of (3), but

$$C_4 e^{m_1 t} + C_5 t e^{m_1 t} + C_6 t^2 e^{m_1 t}$$

is also a solution for properly determined coefficient matrices C_4, C_5, and C_6. Moreover, using the result

$$P(D)t^k e^{mt} = P(m)t^k e^{mt} + \binom{k}{1} P'(m)t^{k-1}e^{mt} + \binom{k}{2} P''(m)t^{k-2}e^{mt} + \cdots,$$

we find that the equations that determine C_4, C_5, and C_6 are

$$A(m_1)C_6 = 0, \quad A(m_1)C_5 = -2A'(m_1)C_6, \quad \text{and} \quad A(m_1)C_4 = -A'(m_1)C_5 - A''(m)C_6.$$

In particular, the first of these shows that like C_3 in the case of a double root, C_6 is also proportional to C_1.

Example. For the system

$$(D^2 + D + 8)x_1 + (D^2 + 6D + 3)x_2 = 0$$
$$(D + 1)x_1 + (D^2 + 1)x_2 = 0$$

we have the characteristic equation $m^4 + 2m^2 - 8m + 5 = 0$ whose roots are $1, 1, -1 \pm 2i$. For the root $-1 + 2i$, (5) becomes

$$\begin{Vmatrix} (4 - 2i) & (-6 + 8i) \\ (2i) & (-2 - 4i) \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix},$$

which is equivalent to the single scalar equation

$$2ic_1 + (-2 - 4i)c_2 = 0.$$

Hence we may take

$$C = \left\| \begin{matrix} c_1 \\ c_2 \end{matrix} \right\| = \left\| \begin{matrix} 1 + 2i \\ i \end{matrix} \right\|.$$

Then from (7) we obtain immediately the two particular solutions

$$X_1 = \left\| \begin{matrix} e^{-t}(\cos 2t - 2 \sin 2t) \\ -e^{-t} \sin 2t \end{matrix} \right\| \quad \text{and} \quad X_2 = \left\| \begin{matrix} e^{-t}(2 \cos 2t + \sin 2t) \\ e^{-t} \cos 2t \end{matrix} \right\|.$$

For the repeated root 1 we have one solution of the form C_1e^t where, from (5)

$$\left\| \begin{matrix} 10 & 10 \\ 2 & 2 \end{matrix} \right\| \cdot C_1 = 0, \text{ so that we may take } C_1 = \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\|.$$

As a second independent particular solution we have

$$^*C_2e^t + C_1te^t$$

where, from (8), $^*C_2$ is any particular solution of the system

$$\left\| \begin{matrix} 10 & 10 \\ 2 & 2 \end{matrix} \right\| \cdot C_2 = - \left\| \begin{matrix} 3 & 8 \\ 1 & 2 \end{matrix} \right\| \cdot C_1 \equiv \left\| \begin{matrix} 5 \\ 1 \end{matrix} \right\|.$$

Hence we may take

$$^*C_2 = \left\| \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right\|.$$

Thus for the double root $m = 1$ we have the two particular solutions

$$X_3 = \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| e^t \equiv \left\| \begin{matrix} e^t \\ -e^t \end{matrix} \right\| \quad \text{and} \quad X_4 = \left\| \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right\| e^t + \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| te^t \equiv \left\| \begin{matrix} te^t \\ \frac{1}{2}e^t - te^t \end{matrix} \right\|.$$

To complete the solution of the nonhomogenous equation (2) it is now necessary to find a particular integral. When the elements of F are functions having only a finite number of independent derivatives the method of undetermined coefficients can be employed very much as in the case of a single equation. For instance, corresponding to matrices F of the forms

$$B,\ Be^{at},\ B_1 \sin at + B_2 \cos at,\ B_1 + B_2t + \cdots + B_nt^n$$

we would assume trial particular integrals of the respective forms

$$K,\ Ke^{at},\ K_1 \sin at + K_2 \cos at,\ K_1 + K_2t + \cdots + K_nt^n.$$

Substituting these into (2) and equating coefficients we obtain systems of linear algebraic equations from which the elements of the coefficient matrices can be found at once provided no term, P_1, in the normal choice of a trial particular integral P duplicates a term in the complementary function. Should such duplication occur, then as in the case of a single equation, the usual choice of a particular integral must be modified. However for a system of equations not only must we multiply by t all those terms in P arising from the terms of F associated with P_1, but we must also retain the duplicating terms as well. Thus if Be^{at} is a term of F which is duplicated by a term of the complementary function, the appropriate choice for a particular integral corresponding to such a term is not just Kte^{at} but rather

$$K_1e^{at} + K_2te^{at}.$$

In this case substitution in (2) yields the equation

$$A(a)K_1e^{at} + A(a)K_2te^{at} + A'(a)K_2e^{at} = Be^{at}.$$

Hence we must have

$$A(a)K_2 = 0 \quad \text{and} \quad A(a)K_1 = B - A'(a)K_2.$$

In particular, the first of these shows that the matrix K_2 is proportional to the matrix coefficient of e^{at} in the complementary function.

The method of variation of parameters also carries over in very much the same fashion to systems of linear differential equations. To employ it, we interpret the scalar coefficients in the general solution of the homogeneous equation (3),

$$X = b_1X_1 + b_2X_2 + \cdots + b_kX_k,$$

not as constants but as functions of t, and attempt to determine them so that X will satisfy the nonhomogeneous system (2). To explore the matter further let (2) be written in the form

$$(A_0D^r + A_1D^{r-1} + \cdots + A_{r-1}D + A_r)X = F. \tag{9}$$

Clearly, in order to substitute X into this equation we will need to know its first r derivatives. Differentiating successively and, as in the case of a single equation, simplifying at each of the first $(r-1)$ stages by equating to zero the terms involving the first derivatives of the b_i we obtain the $(r-1)$ equations:

$$b_1'X_1^{(j)} + b_2'X_2^{(j)} + \cdots + b_k'X_k^{(j)} = 0, \qquad j = 0, 1, 2, \cdots, (r-2). \tag{10}$$

Substituting the various derivatives of X (as simplified) into (9) we obtain

$$\begin{aligned} A_0[(b_1X_1^{(r)} + \cdots + b_kX_k^{(r)}) + (b_1'X_1^{(r-1)} + \cdots + b_k'X_k^{(r-1)})] \\ + A_1[b_1X_1^{(r-1)} + \cdots + b_kX_k^{(r-1)}] + \cdots + A_{r-1}[b_1X_1' + \cdots + b_kX_k'] \\ + A_r[b_1X_1 + \cdots + b_kX_k] = F. \end{aligned}$$

Since each X_i is a solution of the homogeneous system, the last equation reduces to

$$A_0[b_1'X_1^{(r-1)} + \cdots + b_k'X_k^{(r-1)}] = F. \tag{11}$$

Equations (10) and (11) are in effect a system of linear equations from which the b_i can be found without difficulty provided the rank of the augmented matrix of the subsystem (11) is equal to the rank of the matrix of coefficients of this system. This will always be the case if the matrix A_0 is nonsingular. If A_0 is singular then in general the method of variation of parameters cannot be applied directly to (2), though it can be used on any nonsingular system of differential equations to which (2) can be reduced. Of course when the b_i' are determined, the b_i themselves can be found at once by integration.

Example. For the system

$$(D^2 - 7)x_1 - (D^2 - 3)x_2 = f_1$$
$$(D^2 + 2)x_1 + (D^2 - 2)x_2 = f_2$$

we find easily the complementary function

$$X = b_1\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2\begin{Vmatrix} e^{-t} \\ 3e^{-t} \end{Vmatrix} + b_3\begin{Vmatrix} e^{2t} \\ -3e^{2t} \end{Vmatrix} + b_4\begin{Vmatrix} e^{-2t} \\ -3e^{-2t} \end{Vmatrix}.$$

The method of variation of parameters now yields the two matrix equations

$$b_1'\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2'\begin{Vmatrix} e^{-t} \\ 3e^{-t} \end{Vmatrix} + b_3'\begin{Vmatrix} e^{2t} \\ -3e^{2t} \end{Vmatrix} + b_4'\begin{Vmatrix} e^{-2t} \\ -3e^{-2t} \end{Vmatrix} = 0$$

$$\begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix}\left\{b_1'\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2'\begin{Vmatrix} -e^{-t} \\ -3e^{-t} \end{Vmatrix} + b_3'\begin{Vmatrix} 2e^{2t} \\ -6e^{2t} \end{Vmatrix} + b_4'\begin{Vmatrix} -2e^{-2t} \\ 6e^{-2t} \end{Vmatrix}\right\} = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix}.$$

These are equivalent to the four scalar equations

$$e^tb_1' + e^{-t}b_2' + e^{2t}b_3' + e^{-2t}b_4' = 0$$
$$3e^tb_1' + 3e^{-t}b_2' - 3e^{2t}b_3' - 3e^{-2t}b_4' = 0$$
$$-2e^tb_1' + 2e^{-t}b_2' + 8e^{2t}b_3' - 8e^{-2t}b_4' = f_1$$
$$4e^tb_1' - 4e^{-t}b_2' - 4e^{2t}b_3' + 4e^{-2t}b_4' = f_2,$$

from which we find without difficulty

$$b_1 = \int \frac{f_1 + 2f_2}{12}e^{-t}dt, \quad b_2 = -\int \frac{f_1 + 2f_2}{12}e^tdt,$$

$$b_3 = \int \frac{2f_1 + f_2}{24}e^{-2t}dt, \quad b_4 = -\int \frac{2f_1 + f_2}{24}e^{2t}dt.$$

MATHEMATICAL NOTES

EDITED by F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A NOTE ON NORMAL MATRICES

L. MIRSKY, University of Sheffield, England

A matrix is said to be *normal* if it commutes with its transposed conjugate. The product of two normal matrices is not necessarily normal but the following result, proved some years ago by Wiegmann [1], is available.

THEOREM. *If A, B, and AB are normal matrices, then BA is also normal.*

The aim of the present note is to give a very simple proof of Wiegmann's theorem. We denote by X^* the transposed conjugate of the matrix X, and write

$$f(X) = \text{trace } (X^*X), \qquad g(X) = |\lambda_1|^2 + \cdots + |\lambda_n|^2,$$

where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of X. Our proof is based upon the following well-known criterion of Schur [2]:

(1) X is normal if and only if $f(X) = g(X)$.

Since A, B are normal matrices and since, moreover,

$$\text{trace } (XY) = \text{trace } (YX),$$

we have

$$\begin{aligned} f(AB) &= \text{trace } (B^*A^*AB) = \text{trace } (BB^*A^*A) \\ &= \text{trace } (B^*BAA^*) = \text{trace } (A^*B^*BA), \end{aligned}$$

so that

(2) $$f(AB) = f(BA).$$

Now, AB and BA have the same characteristic roots. Therefore

(3) $$g(AB) = g(BA).$$

Again, since AB is normal, we have by (1)

(4) $$f(AB) = g(AB).$$

From (2), (3), and (4) we at once infer that $f(BA) = g(BA)$; and the required conclusion now follows by (1).

References

1. N. A. Wiegmann, Normal products of matrices, Duke Math. J., vol. 15, 1948, pp. 633–638.
2. I. Schur, Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, Math. Annalen, vol. 66, 1909, pp. 488–510.

AN APPLICATION OF ESTIMATION THEORY TO DETERMINANTAL INEQUALITIES

J. B. CHASSAN, St. Elizabeth's Hospital, Washington, D.C.

The inequalities presented below are generalizations of the results obtained by the writer in [1]. In [2] Ky Fan and John Todd first present an algebraic proof of the results obtained in [1] by statistical theory, and then, by an extension of their algebraic argument, derive their determinantal inequality. The determinantal inequalities given below are derived by an extension of the statistical proof used in [1], and, except for the case in which $m=2$, the following results are distinct from those given in [2].*

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1m} \\ a_{21} & a_{22} \cdots a_{2m} \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{n1} & a_{n2} \cdots a_{nm} \end{pmatrix}$$

be a matrix of real elements a_{ij} with $n>m\geqq 2$ and such that all square sub-matrices of order m are non-singular; let $G=A'A$ where A' is the transpose of A. Define $A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}$ as a square matrix consisting of m rows of A including the ith row of A, and preserving their order; let $|G|=\det G$ and let $|G_{j1}|$ be the cofactor of g_{j1} in G where $g_{j1}=\sum_{i=1}^{n} a_{ij}a_{i1}$; let $|A_{i1}^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|$ be the cofactor of a_{i1} in $A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}$. Define

$$\lambda_i = \sum_{\substack{\alpha_1<\alpha_2\cdots<\alpha_{m-1} \\ (\alpha_r\neq i;\, r=1,\cdots,m-1)}}^{n-m+1} \cdots \sum^{n} \frac{|A_{i1}^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|}{|A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|}$$

where each λ_i consists of $C_{n-1,m-1}$ terms.
Then

$$\frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq C_{n,m}^{-2}\sum_{i=1}^{n}\lambda_i^2. \tag{1}$$

Proof: Let $\epsilon_i=z_i-a_{i1}\xi_1-a_{i2}\xi_2-\cdots-a_{im}\xi_m (i=1, 2, \cdots, n)$ represent n random independent deviates with $E(\epsilon_i)=0$, $E(\epsilon_i^2)=1$, random components z_i, known parameters a_{ij} subject only to the restrictions stated above, and with unknown parameters $\xi=(\xi_1, \xi_2, \cdots, \xi_m)$.

The classical least-squares estimate of the vector ξ is determined by selecting the point $\hat{\xi}=(\hat{\xi}_1, \hat{\xi}_2, \cdots, \hat{\xi}_m)$ which minimizes the sum of squares from it to

* For the case of $m=2$, the inequalities stated herein as (1) and (3) are each equivalent to Theorem 1 in the Fan-Todd results, while (2) and (4) are equivalent to their Theorem 2, with the exception that the results given in the Fan-Todd paper extend to the unitary n-space U^n, while the derivations in this paper limit the results to the Euclidean hyperspace.

each of the respective hyperplanes $\epsilon_i = 0$. Each value, $\hat{\xi}_l$, thus obtained is an unbiased estimate of the corresponding parameter component ξ_l and in accordance with the generalized Markoff theorem on least squares [3] each component of $\hat{\xi}$ is the minimum variance unbiased estimate of the corresponding component of ξ in the class of unbiased linear estimates. This holds in particular for $\hat{\xi}_1$ and its variance is well known to be identical to the left side of the inequality stated above [4].

To outline first the remainder of the proof, consider the point of intersection of any m of the n hyperplanes $\epsilon_i = 0$. The coordinates of this point yield unbiased estimates of the corresponding coordinates of the parameter point, ξ. It follows that the arithmetic average of corresponding coordinates of each of the possible $C_{n,m}$ intersections also provides an unbiased estimate of ξ. If this average is labelled $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m)$, then by the aforementioned minimum variance property of $\hat{\xi}$ it follows that $\sigma^2(\hat{\xi}_i) \leqq \sigma^2(\bar{x}_i)$ for all i, and therefore $\sigma^2(\hat{\xi}_1) \leqq \sigma^2(\bar{x}_1)$. It will be shown that

$$\sigma^2(\bar{x}_1) = C_{n,m}^{-2} \sum_{i=1}^{n} \lambda_i^2.$$

To show that

$$x_1^{(1)} = \frac{\begin{vmatrix} z_1 & a_{12} & \cdots & a_{1m} \\ z_2 & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_m & a_{m2} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}}$$

is an unbiased estimate of $\hat{\xi}_1$, we note that

$$E(x_1^{(1)}) = \frac{\begin{vmatrix} E(z_1) & a_{12} & \cdots & a_{1m} \\ E(z_2) & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ E(z_n) & a_{m2} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}};$$

from $\epsilon_i = z_i - a_{i1}\xi_1 - \cdots - a_{im}\xi_m$, and $E(\epsilon_i) = 0$ it follows that $E(z_i) = a_{i1}\xi_1 + a_{i2}\xi_2 + \cdots + a_{im}\xi_m$, and therefore

$$E(x_1^{(1)}) = \frac{\begin{vmatrix} a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1m}\xi_m & a_{12} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}\xi_1 + a_{m2}\xi_2 + \cdots + a_{mm}\xi_m & a_{m2} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}} = \xi_1.$$

Thus $x_1^{(1)}$ is an unbiased estimate of ξ_1 and it follows that $\bar{x}_1 = C_{n,m}\sum_j x_1^{(j)}$ also is.

Next, denote the square sub-matrix consisting of rows $r_1, r_2, \cdots, r_m$ of A by $A^{(r_1, r_2, \cdots, r_m)} \equiv A^{(r)}$. Corresponding to each of the $C_{n,m}$ sets of such submatrices we obtain from $A^{(r)}x^{(r)} \equiv z^{(r)}$,

$$x_1^{(r)} = x_1^{(r_1, r_2, \cdots, r_m)} = \frac{\begin{vmatrix} z_{r_1} & a_{r_1 2} & \cdots & a_{r_1 m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_{r_m} & a_{r_m 2} & \cdots & a_{r_m m} \end{vmatrix}}{\left| A^{(r)} \right|}$$

$$= \frac{1}{\left| A^{(r)} \right|}\left\{ z_{r_1}\left| A_{r_1 1}^{(r)} \right| + z_{r_2}\left| A_{r_2 1}^{(r)} \right| + \cdots + z_{r_m}\left| A_{r_m 1}^{(r)} \right| \right\},$$

where $\left| A_{r_j 1}^{(r)} \right|$ is the cofactor of $a_{r_j 1}$ in $A^{(r)}$. Then the coefficient of z_i in

$$\sum_r x_1^{(r)} = \sum_{r_1 < r_2 \cdots < r_n} \cdots \sum x_1^{(r_1, r_2, \cdots, r_m)}$$

becomes

$$\lambda_i = \sum_{\substack{\alpha_1 < \alpha_2 \cdots < \alpha_{m-1} \\ (\alpha_r \neq i;\, r = 1, \cdots, m-1)}}^{n-m+1} \cdots \sum^{n} \frac{\left| A_{i1}^{(\alpha_1, \alpha_2, \cdots, i, \cdots, \alpha_{m-2}, \alpha_{m-1})} \right|}{\left| A^{(\alpha_1, \alpha_2, \cdots, i, \cdots, \alpha_{m-2}, \alpha_{m-1})} \right|}.$$

It follows from the independence of the z_i and their unit variances that

$$\sigma^2\left(\sum x_1^{(r)} \right) = \sum_{i=1}^{n} \lambda_i^2.$$

Therefore

$$\sigma^2(\bar{x}_1) = C_{n,m}^{-2} \sum_{i=1}^{n} \lambda_i^2,$$

thus establishing (1).

To obtain a weighted generalization of (1), let w be a real vector in $C_{n,m}$-space, restricted to the hyperplane $\sum_{\alpha=1}^{C_{n,m}} w^{(\alpha)} = 1$. Then,

$$(2)\qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}a_{ij}|G_{j1}|\right\}^2 \leqq \sum_{i=1}^{n}\left\{\sum_{\alpha_1<\cdots<\alpha_{m-1}}\cdots\sum w^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}\frac{|A_{i1}^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}|}{|A^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}|}\right\}^2.$$

The proof of (2) is identical to that of (1), with the exception that, while the right hand term of the latter represents the variance of a uniformly weighted arithmetic average of unbiased estimates, the right-hand member of the inequality (2) represents the variance of an unbiased weighted average of the same unbiased estimates, so that Markoff's theorem again applies.

The restriction that each m-order determinant from A be non-singular can be dropped with respect to (2), provided at least one such determinant remains non-singular. When this is the case all of the terms with zero denominators in the right side of (2) may be permitted to vanish, provided that the weights of the non-vanishing terms sum to unity.

Partition the $n\times m$ matrix A into a set of r or $r+1$ submatrices, as follows. Let the ith sub-matrix consist of n_i rows and m columns, with $n_i \geqq m$ for $i=1, 2, \cdots, r$. Let

$$A^{(n_1)} \equiv A^{(1,2,\cdots,n_1)}$$

be the submatrix which contains the first n_1 rows of A,

$$A^{(n_2)} \equiv A^{(n_1+1,n_1+2,\cdots,n_1+n_2)}$$

be that which is comprised of the next n_2 rows,

$$\vdots$$

$$A^{(n_r)} \equiv A^{(n_1+n_2+\cdots+n_{r-1}+1,\cdots,n_1+n_2+\cdots+n_{r-1}+n_r)},$$

the rth submatrix, and containing the last n_r rows of A if $\sum_{i=1}^{r}n_i=n$; if $\sum_{i=1}^{r}n_i<n$, then $n_{r+1}<m$ and the partitioning divides A into $r+1$ instead of r submatrices. Define $G^{(n_l)}=A^{(n_l)'}A^{(n_l)}$ for $l=1, 2, \cdots, r$. Then

$$(3)\qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}a_{ij}|G_{j1}|\right\}^2 \leqq \left(\frac{1}{r}\right)^2\sum_{l=1}^{r}\left[\frac{1}{|G^{(n_l)}|^2}\sum_{i=n_{l-1}+1}^{N_l}\left\{\sum_{j=1}^{m}a_{ij}|G_{j1}^{(n_l)}|\right\}^2\right]$$

where

$$N_k=\sum_{i=1}^{k}n_i.$$

As proof of (3) it need only be stated that the right side represents the variance

of a uniformly weighted arithmetic average of r unbiased least squares estimates of ξ_1.

As a weighted generalization of (3) there follows, subject to $\sum_{l=1}^{r} w_l = 1$,

$$(4) \qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq \sum_{l=1}^{r}\left[\frac{w_l^2}{|G^{(n_l)}|^2}\sum_{i=N_{l-1}+1}^{N_l}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}^{(n_l)}|\right\}^2\right].$$

In this case the greater side is the variance of a weighted average of unbiased least squares estimates.

References

1. J. B. Chassan, A statistical derivation of a pair of trigonometric inequalities, this MONTHLY, vol. 62, 1955, pp. 353–356.

2. Ky Fan and John Todd, A determinantal inequality, Jour. of the London Math. Soc., vol. 30, 1955, pp. 58–64.

3. J. Neyman and F. N. David, Extension of the Markoff Theorem on least squares, Statistical Research Memoirs, II, p. 105.

4. Whittaker and Robinson, The Calculus of Observations, 4th Edition, Chapter IX, London, Blackie and Sons, 1940.

$\Gamma(z)$ AT MINUS INFINITY

W. FULKS, University of Minnesota

The behavior of $\Gamma(z)$ in the distant portions of the complex plane outside any fixed angle containing the negative real axis is given by Stirling's formula

$$(1) \qquad \Gamma(z) \sim \sqrt{2\pi}z^{z-1/2}e^{-z},$$

which is valid as $|z|\to\infty$, $-\pi+\delta\leqq\arg z\leqq\pi-\delta$ for any δ, $0<\delta<\pi$. That the behavior inside the exceptional angle follows from this formula and the identity

$$(2) \qquad \Gamma(z)\cdot\Gamma(1-z) = \pi \csc \pi z$$

seems not to be widely known. Let z be any fixed complex number, not an integer. Then $(z-n)$, for large integer n, is in the critical angle, but bounded away from the poles. Furthermore $(1-z+n)$ is exterior to this angle so that (1) applies and we have, as n tends toward infinity:

$$\begin{aligned}\Gamma(z-n) &= \frac{\pi \csc \pi(z-n)}{\Gamma(1+n-z)} = \frac{(-1)^n\pi \csc \pi z}{\Gamma(1+n-z)} \\ &\sim \frac{(-1)^n\pi \csc \pi z}{\sqrt{2\pi}}(1+n-z)^{z-n-1/2}e^{1+n-z} \\ &\sim (-1)^n\sqrt{\frac{\pi}{2}}\,n^{z-n-1/2}e^n \csc \pi z.\end{aligned}$$

This formula can, of course, be improved by using more terms in the complete asymptotic expansion of which (1) gives the first term. More labor would then yield more terms in the resulting asymptotic expression for $\Gamma(z-n)$.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON TABLES OF FACTORS AND PRIMES

LEONARD CANERS, St. Michael's College

The object of this paper is to describe a method making use of "key numbers and possible factors" which will increase the usefulness of any table of factors and primes approximately tenfold.* This method will enable one, by inspection alone, to determine whether a given number up to about 10 times the largest entry given in a table of primes is prime or not, and if not prime what its factors are.

The following symbols will be used:

N is the number whose factors, if any, are to be determined.

A is N with the last digit dropped. Thus if $N=17{,}873$, $A=1787$.

R is the range and is the integral part of the square root of N with its last two digits dropped. Thus R of $17{,}873=13$

K is the key number and is found in the table below.

P_1 is a possible factor corresponding to K.

The twin series are given in the table for the sake of completeness but are not written down in actual practice.

There are two steps as outlined below.

STEP I

N	K	P_1	Series	Procedure
Any number ending in 1, 3, 7, or 9 within limits stated above	$A-R$	$10R+$last digit	$K+n$; P_1-10n $n=0, 1, \cdots, 2R+1$	Beginning with K read Table of Factors and Primes from left to right and look for corresponding possible factors.

STEP II

N ending in	K	P_1	Series	Procedure
1	$(A-2)-3R$	$10R+7$	$K+3n$; P_1-10n $n=0, 1, \cdots, 2R+1$	Same as in Step I except that only every *third* entry in Table of Primes and Factors is examined for corresponding possible factors.
3	$A-3R$	$10R+1$	$n=0, 1, \cdots, 2R+1$	
7	$(A-2)-3R$	$10R+9$	$n=0, 1, \cdots, 2R+1$	
9	$A-3R$	$10R+3$	$n=0, 1, \cdots, 2R+1$	

* A table of factors and primes having 2009 entries will suffice to determine any number up to 19,949.

As an example consider 7519. We shall use mathematical tables from the *Handbook of Chemistry and Physics.*

Step I

From our table $N=7519$; $A=751$; $R=8$; $K=743$; $P_1=89$ and the series obtained:

743	89
744	79
745	69
746	59
—	—
751	9
752	1
753	11
—	—
760	81

Examining the Table of Factors and Primes beginning with 743 we note whether 743 has 89 as a factor; whether 744 has 79 as a factor, *etc.* We continue to read from left to right keeping in mind the possible factors 69, 59 $\cdots$ 81.

Since none are found we conclude that 7519 has no factors within the given range ending in 9 or 1.

Step II

From our table $K=727$ and $P_1=83$; the series obtained is:

727	83
730	73
733	63
—	—
751	3
754	7
—	—
778	87

We proceed exactly as in Step I. Beginning with 727 we note whether it has 83 as a factor. Thereafter we examine every *third* entry for the remaining corresponding possible factors 73, 63 $\cdots$ 3, 7, $\cdots$ 87. Since 730 yields the factor 73 we conclude that 73 is a factor of 7519. By division $7519=73\times103$. Had no factor been found in either Step I or Step II we would have concluded that the number under consideration was prime.

Proof for Step I

Let N end in 1. Then, if N has a factor ending in 1, we can write

$$(10a+1)(10b+1)=N=10A+1,$$

$$10ab + a + b = A$$

or

$$b(10a + 1) = (A - a).$$

This proves that any factor $10a+1$ of N is also a factor of $A-a$.

If we put $a=(R-n)$, then $(10a+1)$ becomes $(10R+1)-10n$ and $(A-a)$ becomes $(A-R)+n$, *i.e.*, P_1-10n and $K+n$ respectively as given in the table. Since P_1-10n becomes a number ending in 9 when $P_1<10n$, all possible factors ending in 9 are also provided for.

The proof for Step II is similar. For possible factors ending in 3 or 7 we write

$$(10a + 7)(10b + 3) = N = 10A + 1.$$

By identical reasoning this results in P_1-10n and $K+3n$ respectively given in the table. This completes the proof for numbers ending in 1.

Identical reasoning establishes the key numbers and series for numbers ending in 3, 7, or 9.

A METHOD FOR SOLVING SCHLÖMILCH'S INTEGRAL EQUATION

JOHN R. HATCHER, Fisk University

In chapter 11 of the fourth edition of their work *A Course of Modern Analysis*, Whittaker and Watson devote sections 11.8–11.81 to the integral equations of Abel and Schlömilch. After reading this reference, however, the student is not likely to observe the fact that these equations are closely related. This is perhaps due to the manner in which the equations are solved. In this note we offer an alternative method for the solution of Schlömilch's equation

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} g(x \sin \theta)d\theta, \qquad f, g \in C'([-\pi, \pi]) \tag{1}$$

which not only seems simpler and more direct, but, as a consequence of Abel's equation, gives what is believed to be a new expression for the solution.

The substitutions $x \sin \theta = \sqrt{\xi}$, $x^2 = t$ readily transform (1) to the ordinary Abel equation

$$f(\sqrt{t}) = \int_0^t (t - \xi)^{-1/2} G(\xi)d\xi, \tag{2}$$

where $G(\xi)$ denotes the function $\pi^{-1}\xi^{-1/2}g(\sqrt{\xi})$. But the solution of (2) is

$$G(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t (t - \xi)^{-1/2} f(\sqrt{\xi})d\xi, \tag{3}$$

and therefore

$$g(x) = x\frac{dx}{dt}\frac{d}{dx}\int_0^{\pi/2}(x^2 - x^2\sin^2\theta)^{-1/2}f(x\sin\theta)x^2\sin 2\theta\,d\theta$$

(4)

$$= \frac{d}{dx}\int_0^{\pi/2}(x\sin\theta)f(x\sin\theta)d\theta$$

or equivalently,

$$g(x) = f(0) + x\int_0^{\pi/2} f'(x\sin\theta)d\theta. \tag{5}$$

This follows from (4) by differentiating the integral and then integrating by parts. For

$$g(x) = \int_0^{\pi/2}[\sin\theta f(x\sin\theta) + x\sin^2\theta f'(x\sin\theta)]d\theta$$

$$= -f(x\sin\theta)\cos\theta\Big|_0^{\pi/2} + x\int_0^{\pi/2}\cos^2\theta f'(x\sin\theta)d\theta$$

(6)

$$+ x\int_0^{\pi/2}\sin^2\theta f'(x\sin\theta)d\theta$$

$$= f(0) + x\int_0^{\pi/2} f'(x\sin\theta)d\theta.$$

DERIVING THE EQUATIONS OF THE SECTIONS OF A CONE

LUISE LANGE, Chicago Teachers College

In a recent note G. B. Huff* called attention to the gap which the traditional presentation in Analytic Geometry leaves between the two sets of definitions of the "conic sections," as plane sections of a right circular cone on the one side, and as the several geometrical loci on the other. He closed this gap by showing geometrically (or drawing attention to proofs given elsewhere) that the variously inclined sections of a cone satisfy the respective locus conditions.

In the following the equivalence of the two sets of definitions is demonstrated analytically. That is, we show—without using analytic space geometry—that the equations of the curves of intersection of a right circular cone with variously inclined planes are the same as those obtained from the locus definitions; the coefficients moreover reveal the relation between the parameters characterizing the sections of the cone and those characterizing the loci.

Let VAB (Fig. 1) be a section through the axis of a right circular cone, and Γ a plane perpendicular to the axis at C, intersecting the cone in a circle of radius r. Let α be the angle between the elements of the cone and Γ.

Intersect the cone with a second plane Δ through C forming the dihedral angle $\phi = ACD$ with Γ. On the intersection of Δ with the cone take an arbitrary

* G. B. Huff, On defining conic sections, this MONTHLY, vol. 62, 1955, pp. 250–251.

point P. Draw $CP=\rho$, forming angle $\theta=DCP$ with CD. Then (ρ,θ) are the polar coordinates of P in plane Δ, with C as origin and the line through CD as polar axis.

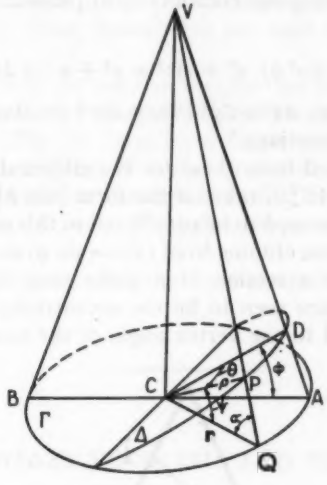

FIG. 1

Draw the element VP intersecting the circle in plane Γ at Q. Connect Q with C. Then the angle $\psi=PCQ$, which ρ forms with plane Γ, is given by the relation

$$\sin\psi = \sin\phi\cdot\cos\theta. \tag{1}$$

Plane Δ is inclined towards Γ by angle ϕ; CP forms angle θ with the "line of steepest ascent" CD; ψ is the inclination of CP towards Γ. This leads readily to (1). (See Fig. 2.)

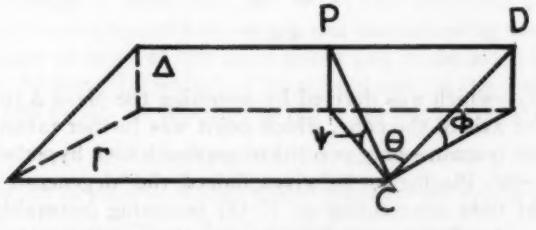

FIG. 2

We next derive the relation between ρ and θ from triangle CPQ in which $CP=\rho$; $CQ=r$; angle $Q=\alpha$; angle $C=\psi$. Hence $\rho/\sin\alpha=r/\sin(\alpha+\psi)$. Expanding $\sin(\alpha+\psi)$, and expressing $\sin\psi$ and $\cos\psi$ in terms of ϕ and θ by virtue of (1),

one obtains the polar equation of the curve of intersection:

$$(2) \quad f(\rho, \theta) = (\sin^2 \alpha - \sin^2 \phi \cdot \cos^2 \theta) \cdot \rho^2 + r \cdot \sin 2\alpha \cdot \sin \phi \cdot \rho \cos \theta - r^2 \sin^2 \alpha = 0.$$

Transforming to rectangular coordinates in plane Δ, $x = \rho \cdot \cos \theta$, $y = \rho \cdot \sin \theta$, one obtains

$$(3) \quad F(x, y) = (\sin^2 \alpha - \sin^2 \phi) \cdot x^2 + \sin^2 \alpha \cdot y^2 + r \cdot \sin 2\alpha \cdot \sin \phi \cdot x - r^2 \sin^2 \alpha = 0.$$

This equation is the form $Ax^2+Cy^2+Dx+Ey+F=0$ obtained from the locus definitions of the "conic sections."

The conditions derived from these for the elliptical, parabolic, and hyperbolic case respectively, $AC \gtreqless 0$, take on the form $|\sin \phi| \gtreqless |\sin \alpha|$ or $\phi \gtreqless \alpha$ (since only first-quadrant angles need to be considered in this context).

For the eccentricity one obtains from (3) $e = \sin \phi / \sin \alpha$. This shows that the greatest eccentricity for a section of a given cone is obtained for $\phi = 90°$, $e = 1/\sin \alpha$. This is at once seen to be the eccentricity of a hyperbola whose asymptote angle is equal to the vertex angle of the cone.

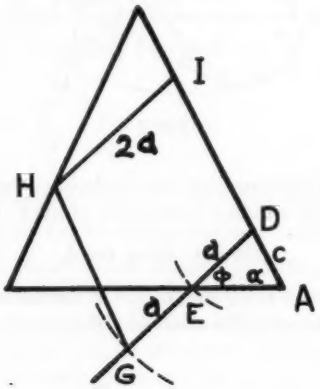

FIG. 3

Equation (3)—which was derived by assuming the plane Δ to pass through a point C on the axis of the cone, which point was further taken as the origin of the coordinate system,—only permits to approach such hyperbolic sections as the limit for $\phi \to 90°$. Placing $\phi = 90°$ gives, indeed, the "degenerate" axial section the two straight lines intersecting at V, (3) becoming factorable in the form $(x \cos \alpha - r \sin \alpha)^2 - y^2 \sin^2 \alpha = 0$. But the equations of the sections parallel to the axis ($\phi = 90°$) are easily derived separately. Passing plane Δ through a point D on the cone parallel to the axis at distance k, one readily obtains for the rectangular coordinates of a point P on the curve of intersection, with respect to D as origin,

$$\frac{(x+h)^2}{h^2} - \frac{y^2}{k^2} = 1,$$

where $h = k \cdot \tan \alpha$, and hence $(-h, -k)$ are the coordinates of vertex V of the cone. (Proof omitted.) These hyperbolas are seen to have the eccentricity $e = 1/\sin \alpha$.

We still point out that the relation $e = c/a = \sin \phi/\sin \alpha$ furnishes a simple construction of a central conic of specified axis $2a$ and focal distance $2c$ to be cut from a given cone (Fig. 3). From an arbitrary point A on the cone lay off $AD = c$ along an element. From D strike an arc of radius a intersecting the diameter of the circular section through A at E. DE then forms with EA the proper angle ϕ. Construct $DG = 2a$. The indicated parallel construction gives the specified axis $HI = 2a$.

This construction again implies the limiting condition, $a \geqq c \cdot \sin \alpha$, that is, $e \leqq 1/\sin \alpha$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1226. *Proposed by J. V. Pennington, Houston, Texas*

A rancher bought a white cow, and in the following year a red one. Each succeeding year he duplicated his purchases of the preceding two years, buying the same number of cows, of the same colors and in the same order. Thus, in the third year, he bought a white and then a red cow; in the fourth year, a red, then a white, and then a red cow; and so on. What was the color of the nth cow?

E 1227. *Proposed by R. L. Helmbold, Carnegie Institute of Technology*

Find all values of $a \geqq 1$ such that $a^x \geqq x^a$ holds for all values $x \geqq 0$.

E 1228. *Proposed by Viktors Linis, University of Ottawa*

Let $n(P)$ be the number of distinct lines L through a point P such that L divides the area of a given triangle in two equal parts. Show that the locus of all points P with $n(P) \geqq 2$ is a region the ratio of whose area to the area of the given triangle is an absolute constant.

E 1229. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

Given any point O in the plane of a triangle $\Delta \equiv ABC$, let the sides a, b, c of Δ subtend angles A', B', C' at O, and let the distances from O to the vertices of Δ be a', b', c'. Show that the triangle with sides aa', bb', cc' has angles $A'-A$, $B'-B$, $C'-C$, and find the sextic polynomial relation connecting a, b, c, a', b', c'.

E 1230. *Proposed by Vassili Daiev, Sea Cliff, Long Island*

Find the nth term of the sequence of ordered positive integers prime to 30.

SOLUTIONS

Permutations and Permutable Transpositions

E 1118 [1954, 345; 1955, 43]. *Proposed by Joel Brenner, State College of Washington*

Let us say a permutation is of type T if it is a product of mutually permutable transpositions. Is every permutation p a product of two permutations of type T?

Addendum by the Proposer. Every permutation is a product of disjoint cycles. In [1955, 43], a proof was given for finite cycles. For infinite cycles the answer is also yes, for

$$(\cdots, -4, -3, -2, -1, 0, 1, 2, 3, \cdots) = R \cdot S,$$

where

$$R = (0, -1)(1, -2)(2, -3)\cdots, \qquad S = (-1, 1)(-2, 2)(-3, 3)\cdots.$$

The 2319th Digit in 1000!

E 1180 [1955, 493; 1956, 189]. *Proposed by M. S. Klamkin and Alex Kraus, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

Editorial Note. H. S. Uhler has calculated the exact value of 1000!. See his article "Exact values of 996!, and 1000!, with skeleton tables of antecedent constants," *Scripta Mathematica*, vol. XXI, pp. 261–268. The 2319th digit in the expansion of 1000! appears as 2, as was shown in [1956, 189].

An Incomplete Assignment

E 1196 [1956, 39]. *Proposed by T. S. Ferguson and C. H. Kraft, University of California*

A student awoke at the end of a class in algebra one morning just in time to hear his teacher say, " . . . and I will give you the hint that all the roots are real and positive." Looking at the blackboard he discovered a 20th degree equation to be solved for home work, which he hastily tried to copy down. He succeeded in getting only the first two terms, $x^{20}-20x^{19}$, before his teacher erased the board completely; however, he did remember that the constant term was $+1$. Can you help our hero by not only solving the equation, but also showing uniqueness

of the solution?

Solution by D. S. Greenstein, University of Pennsylvania. The roots are positive and have their arithmetic and geometric means both equal to 1. It follows that the roots are all equal to 1.

Also solved by W. E. Baxter, G. E. Bills, Louis Brickman, P. L. Chessin, A. E. Danese, J. M. Elkin, Michael Goldberg, A. J. Goldman, Virginia Hanly, Vern Hoggatt, F. A. Homann, A. R. Hyde, Lawrence Isenecker, Edgar Karst, M. S. Klamkin, D. C. B. Marsh, Morris Morduchow, J. B. Muskat, Herbert Nadler, Walter Penney, L. L. Pennisi, C. F. Pinzka, M. A. Rashid, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., E. D. Schell, J. P. Scholz, R. R. Seeber, Jr., R. E. Shafer, O. E. Stanaitis, Art Steger, Chih-yi Wang, David Zeitlin, and the proposers. Late solution by Alan Wayne.

Editorial Note. For a proof of the fact that the arithmetic and geometric means of n positive numbers are equal if and only if the n numbers are all equal, see *e.g.*, G. Chrystal, *Algebra*, Part II, pp. 46–47, or Hardy, Littlewood and Pólya, *Inequalities*, p. 17. As was pointed out by Wang, if the word "positive' is omitted from the statement of the problem, then uniqueness cannot be shown. Thus we might also take the roots to be eight 2's, four 1's, four 1/2's, and four $-1/2$'s.

A Rich Configuration

E 1197 [1956, 39]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be a right triangle and CH the altitude on the hypotenuse AB. Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH.

Solution by Leon Bankoff, Los Angeles, Calif. I. The diameter of the circle inscribed in a right triangle is equal to the sum of the legs minus the hypotenuse. Applying this relation in triangles ABC, HCA, HCB, we get

$$\frac{(AC + CB - AB) + (AH + CH - AC) + (CH + HB - CB)}{2} = CH.$$

II. In similar right triangles, the ratios of inradius to hypotenuse are equal. We may therefore write

$$r/c = r_1/b = r_2/a = (r + r_1 + r_2)/(a + b + c),$$

where r, r_1, r_2 are the inradii of triangles ABC, HCA, HCB. Since

$$r/c = CH/(a + b + c)$$

it follows that $r + r_1 + r_2 = CH$.

Additional selected properties of the configuration. R, S, T are the incenters of triangles AHC, CHB, ABC, respectively, and H, R', S', T' the orthogonal projections of C, R, S, T on AB. Let r, r_1, r_2 denote the inradii of triangles ABC,

AHC, CHB. P and Q are the feet of the cevians CR and CS. RS cuts AC in U and CB in V. K is the intersection of PS and RQ.

(1) T' is the circumcenter of triangle RST; Q, S, T, R, P are concyclic; $PT'=TT'=T'Q$.

(2) T is the orthocenter of triangle CRS and the circumcenter of triangle CPQ.

(3) $r_1^2+r_2^2=r^2$.

(4) $RS=CT=PT=QT=r\sqrt{2}$.

(5) Triangles HSR, ABC, AHC, HCB are similar.

(6) A, R, S, B are concyclic. Triangles RST and ABT are inversely similar.

(7) R, T', H, S are concyclic. (RS is a diameter of the circle.)

(8) $T'S$ is parallel to AC; RT' is parallel to CB.

(9) Triangles $RR'T'$ and $T'S'S$ are congruent (and similar to triangle ABC).

(10) Triangles ATB, BCS, ARC are similar.

(11) Angle ATB = angle BSC = angle ARC = 135°.

(12) $AC=AQ$; $PB=CB$.

(13) $CU=CV$; CT is the perpendicular bisector of UV.

(14) PS, RQ, CH are concurrent at K, the orthocenter of triangle CPQ.

(15) PS is parallel to AT; RQ is parallel to TB; triangles PBS and CSB are congruent; triangles AQR and ARC are congruent.

(16) The midpoint of RS is the nine point center of triangle CPQ.

(17) The circumcircle of triangle HSR is the nine point circle of triangle CPQ.

(18) The circumcircles of triangles ARC and CSB are tangent at C, and CT is their common internal tangent.

(19) $RT=KS=SQ$; $RP=RK=TS$; triangles PKR and KQS are isosceles right triangles. (Also triangle $RT'S$.)

(20) A, P, T, C are concyclic; B, Q, T, C are concyclic.

(21) The perimeter of triangle $T'S'S$ = perimeter of triangle $RR'T=CH$ (since $SS'=r_2, T'S'=r_1$, $T'S=r$).

(22) Area of triangle $RST=(a+b-c)^3/8c=r^3/c$.

(23) Area of pentagon $PQSTR=2r^3/c+r^2$.

(24) Area of triangle $CPQ=abr/c$.

Also solved by W. A. Al-Salam, L. C. Barrett, Robert Bart, G. E. Bills, R. L. Caskey, G. B. Charlesworth, N. A. Childress, T. Y. Chow, Mary Constable, R. J. Cormier, K. W. Crain, A. E. Danese, D. E. D'Atri, G. W. Day, Hazel Evans, Herta Freitag, Michael Goldberg, A. J. Goldman, Peter Gould, Cornelius Groenewoud, D. J. Hansen, Vern Hoggatt, R. T. Hood, Roger Hou, J. P. Hoyt, Raymond Huck, Louise Hutchinson, A. R. Hyde, P. W. M. John, Edgar Karst, M. S. Klamkin, W. G. Koellner, Sam Kravitz, M. A. Laframboise, L. E. Laird, B. R. Leeds, L. I. Lokomowitz, Robert Lynch, D. C. B. Marsh, Beckham Martin, C. N. Mills, C. S. Ogilvy, Margaret Olmsted, M. J. Pascual, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), C. F. Pinzka, P. W. A. Raine, M. A. Rachid, L. A. Ringenberg, Azriel Rosenfeld, Donald Rubin, C. M.

Sandwick, Sr., E. D. Schell, G. J. Simmons, Bernard Smilowitz, Sister M. Stephanie, A. V. Sylwester, W. R. Talbot, Chih-yi Wang, R. M. Warten, Dale Woods, Roscoe Woods, André Yandl, David Zeitlin, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

It was pointed out that this problem appears in N. A. Court, *College Geometry*, 2nd ed., p. 93, ex. 19b, and in *Scripta Mathematica*, vol. 16 (1950), p. 167.

A Coin Tossing Procedure

E 1198 [1956, 39]. *Proposed by E. T. Frankel, U. S. Dept. of Health, Education and Welfare, New York, N. Y.*

Describe a coin tossing procedure to determine a random integer, such that all integers from 1 to n have equal probabilities of being selected.

Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J. A simple but inefficient procedure is to label n coins $1, \cdots, n$ and let the choice of the integer k correspond to the event in which all coins but the kth fall tails.

A more efficient procedure is to toss s labeled coins, where s is the smallest integer such that $2^s \geqq n$, and let $1, \cdots, n$ correspond to n of the 2^s possible outcomes.

Also solved by J. L. Alperin, J. L. Baker, G. B. Charlesworth, Michael Goldberg, Vern Hoggatt, A. R. Hyde, Lawrence Isenecker, P. W. M. John, M. S. Klamkin, L. E. Laird, D. C. B. Marsh, J. B. Muskat, Herbert Nadler, C. S. Ogilvy, M. J. Pascual, Walter Penney, Ronald Pyke and S. C. Saunders (jointly), C. M. Sandwick, Sr., E. D. Schell, G. J. Simmons, A. V. Sylwester, W. R. Van Voorhis, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

Maximum Area Between a Hanging Chain and Its Chord

E 1199 [1956, 39]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

A flexible chain of length L is suspended from its endpoints. Determine the maximum area between the chord joining the endpoints and the hanging curve.

Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pa. Let $-x+w$ and $x+w$ be the abscissas of two points on the curve having equation $y=\cosh x$, such that the figure formed by the arc and the chord joining the two points is similar to the chain of length L and the chord joining its points of suspension. Let A be the plane area bounded by the chain and its chord, and let z be the length of the chord. Then

$$z = L(x^2 + \sinh^2 w \sinh^2 x)^{1/2}/\cosh w \sinh x,$$

$$A = L^2(x \cosh x - \sinh x)/2 \cosh w \sinh^2 x.$$

If w is held constant, $dA/dz=0$ when

$$x = 2(\sinh x \cosh x)/(1 + \cosh^2 x) = 1.6062, \text{ approximately.}$$

For any x, A is a maximum when $w=0$, which occurs when and only when the chord is horizontal. Then

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x \quad \text{and} \quad z = Lx/\sinh x;$$

so the maximum area is approximately $0.1549L^2$, the points of suspension being joined by a horizontal chord whose length is approximately $0.6716L$.

Also solved by Hüseyin Demir, Michael Goldberg, Vern Hoggatt, A. R. Hyde, Lawrence Isenecker, D. C. B. Marsh, J. W. Ross, and Ernst Trost. Late solution by A. J. Goldman.

Editorial Note. Eliminating x from

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x \text{ and } L = 2 \sinh x$$

we find

$$A_{\max} = (L^2 + 4)^{1/2} \sinh^{-1} (L/2) - L.$$

Sum of an Infinite Series

E 1200 [1956, 39]. *Proposed by Sidney Weiner, Washington, D. C.*

Evaluate the convergent infinite series

$$S = \sum_{k=1}^{\infty} \left(\sum_{p=1}^{2k-1} 1/p \right) \Big/ 2k(2k+2).$$

Solution by W. A. Al-Salam, Duke University. We have

$$\begin{aligned} 4S &= \sum_{k=1}^{\infty} \left(\sum_{p=1}^{2k-1} 1/p \right) \Big/ k(k+1) \\ &= \sum_{k=1}^{\infty} 1/k(k+1) + (1/2 + 1/3) \sum_{k=2}^{\infty} 1/k(k+1) \\ &\quad + (1/4 + 1/5) \sum_{k=3}^{\infty} 1/k(k+1) + \cdots . \end{aligned}$$

But

$$\sum_{k=m}^{\infty} 1/k(k+1) = 1/m.$$

Therefore

$$\begin{aligned} 4S &= 1 + \sum_{k=2}^{\infty} [1/(2k-2) + 1/(2k-1)]/k \\ &= 1 + (1/2) \sum_{k=2}^{\infty} 1/k(k-1) + \sum_{k=2}^{\infty} 1/k(2k-1) \end{aligned}$$

$$= 1 + 1/2 + 2\sum_{k=2}^{\infty} [1/(2k-1) - 1/2k]$$
$$= 3/2 + 2[1/3 - 1/4 + 1/5 - 1/6 + \cdots]$$
$$= 3/2 + 2[\ln 2 - (1 - 1/2)]$$
$$= 1/2 + 2 \ln 2$$

and

$$S = 1/8 + (\ln 2)/2.$$

Also solved by G. B. Charlesworth, T. Y. Chow, A. E. Danese, Nathaniel Grossman, Emil Grosswald, Vern Hoggatt, A. R. Hyde, M. S. Klamkin, P. K. Maloof, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, Walter Penney, L. L. Pennisi, M. A. Rashid, L. A. Ringenberg, David Rosen, E. D. Schell, R. E. Shafer, O. E. Stanaitis, Chih-yi Wang, G. N. Wollan, David Zeitlin, and the proposer. Late solutions by A. J. Goldman, Virginia Hanly, and Alan Wayne

---

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4698. *Proposed by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland*

Prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m a_n}{m+n+1/6} \leqq 2\pi \sum_{r=0}^{\infty} a_r^2,$$

provided the right-hand side converges.

4699. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let $f(x)$ be a frequency function of finite variance σ^2 and such that

$f(x)\cdot\log f(x)$ is summable ($x\log x$ is defined to be $=0$ if $x=0$). Show that

$$\int_{-\infty}^{\infty} f(x)\log f(x)dx \geqq \log\frac{1}{\sigma\sqrt{2\pi e}}$$

with equality if and only if $f(x)$ is equal almost everywhere to a normal function

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-m)^2/2\sigma^2}.$$

4700. *Proposed by J. W. Clawson, Ursinus College*

It is well known that the midpoints of the diagonals of a 4-line lie on a straight line which has been called the *Newtonian* of the 4-line. It is also known that the Newtonians of the five 4-lines obtained by omitting in turn each of the sides of a 5-line concur in a point which we may call the *Newtonian point* of the 5-line.

Prove (1) that the Newtonian points of the six 5-lines obtained by omitting in turn each of the sides of a 6-line lie on a conic which may be called the *Newtonian conic* of the 6-line; (2) that the Newtonian conics of the seven 6-lines obtained by omitting in turn each of the sides of a 7-line concur in three points, two of which may be imaginary.

4701. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Let α be complex, $|\alpha|<1$. Prove that there exists a power series $\sum a_k z^k$ with rational integral coefficients, $|a_k|\leqq|1/\alpha^2|$, such that $\sum a_k\alpha^k=0$.

4702. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn., and W. E. Weissblum, Massachusetts Institute of Technology*

Let

$$\begin{matrix} A_{11}, & A_{12}, & A_{13}, \cdots \\ A_{21}, & A_{22}, & A_{23}, \cdots \\ \cdots & \cdots & \cdots \end{matrix}$$

be an infinite square array of sets of real numbers in (0, 1). If $A_{kn}\to 0$ for each fixed k, does it follow that there exists a diagonal sequence

$$A_{1n_1}, \quad A_{2n_2}, \quad A_{3n_3}, \cdots$$

whose limit is 0?

SOLUTIONS

Prime Divisors of Fibonacci Numbers

4637 [1955, 259]. *Proposed by W. E. Briggs, University of Colorado.*

Let $f(n)$ be the smallest number such that $u_{f(n)}$ is divisible by n, where $u_1=1$, and $u_2=1$ and $u_m=u_{m-1}+u_{m-2}$. Prove (or disprove) that for any prime p greater

than 2 and for all n,

$$f(p^n) = p^{n-1}f(p).$$

Solution by V. D. Gokhale, Atlanta University. We first state four lemmas, of which the first three are obvious and the fourth is easily proved by induction on r.
Lemma 1. If $n>1$ divides u_p, it does not divide u_{p-1}.
Lemma 2.

$$u_{h+k} = u_k u_{k+1} + u_{k-1}u_k.$$

Lemma 3. If p^n divides u_k, it divides u_{h+k} if and only if it divides u_h.
Lemma 4.

$$u_{rm} = \sum_{t=0}^{r-1} {}_rC_t u_{r-t} u_m^{r-t} u_{m-1}^{t},$$

$$u_{rm-1} = \sum_{t=0}^{r-1} {}_rC_t u_{r-t-1} u_m^{r-t} u_{m-1}^{t} + u_{m-1}^{r}.$$

Since

$$u_{f(p^{n+1})}$$

is divisible by p^n, it follows from lemma 3 that $f(p^{n+1})$ is a multiple of $f(p^n)$. Set $f(p^{n+1}) = rf(p^n)$. In the first expansion of lemma 4, all terms except the last are divisible by p^n where $m = f(p^n)$. The last term is

$$r u_{f(p^n)-1}^{r-1} u_{f(p)}.$$

Now p does not divide $u_{f(kp)-1}$. Hence if p^λ is the highest power of p which divides $u_{f(p)}$, $p^{\lambda+1}$ divides

$$u_{rf(p^n)}$$

if and only if p divides r. Hence if $f(p^\lambda) = f(p^n)$, then $f(p^{\lambda+1}) = pf(p^n)$. Hence it follows that if $f(p) = f(p^k)$ then $f(p^{k+1}) = pf(p)$ and by a finite number of steps $f(p^{k+n}) = p^n f(p)$. Hence we have proved: *If for a prime $p>2$, $u_{f(p)}$ is not divisible by p^2, then $f(p^n) = p^{n-1}f(p)$ for every n.*

It does not appear whether $u_{f(p)}$ can ever be a multiple of p^2. For $p=2$, the result differs only slightly. We find $f(2)=3$, $f(2^2) = f(2^3) = 6$, $f(2^n) = 2^{n-2}f(2)$ for $n>2$.

Also solved by Leonard Carlitz, M. S. Klamkin, and Chih-yi Wang.

Editorial Note. Klamkin notes that these results were stated by Lucas (*Comptes Rendus*, Paris, 82, 1876, pp. 1303–1305) as being true also of the slightly more general sequence

$$u_n = \frac{a^n - b^n}{a - b},$$

where a and b are roots of a quadratic equation with integral coefficients. The present problem is the case $a=(1+\sqrt{5})/2$, $b=(1-\sqrt{5})/2$.

Identity

4639 [1955, 371]. *Proposed by L. B. Rall, Oregon State College*

Given a set of distinct numbers $a_0, a_1 \cdots$, such that $a_j \neq a_k$ if $j \neq k$, prove that for all positive integers n,

$$\sum_{j=0}^{n} \prod_{k=0, k\neq j}^{n} \frac{(a_k - a_{n+1})}{(a_k - a_j)} = 1.$$

Solution by Leonard Carlitz, Duke University. The stated identity is an immediate special case of the Lagrange interpolation formula (see, *e.g.*, Weisner, *Theory of Equations*, New York, 1938, p. 57 and ex. 4(b), p. 58). However, a direct proof is quite simple. Put

$$L_j(x) = \prod_{k=0, k\neq j}^{n} \frac{a_k - x}{a_k - a_j}.$$

Clearly $L_j(a_j)=1$ while $L_j(a_k)=0$ for $0 \leqq k \leqq n$, $k \neq j$. Consequently

$$\sum_{j=0}^{n} L_j(x) = 1 \tag{1}$$

for $x=a_0, a_1, \cdots, a_n$. The left member of (1) is a polynomial of degree $\leqq n$ which is equal to 1 for $n+1$ distinct values of x. Therefore (1) is an identity.

Also solved by W. J. Blundon, R. H. Breusch, N. J. Fine, A. R. Hyde, D. C. B. Marsh, W. V. Parker, Berthold Schweizer, M. F. Smiley, Chih-yi Wang, and the Proposer. Late solution by M. S. Klamkin.

Subgroups

4640 [1955, 371]. *Proposed by M. L. Keedy, University of Nebraska*

Let H be a proper subgroup of a group G and $\overline{H}$ its complement in G. Show that $H'=\overline{H}\overline{H}$ is a normal subgroup of G, and that it equals either G or H. Characterize those subgroups for which $H'=H$.

Solution by Virginia S. Hanly, Ohio State University. If the index of H in G is 2, then $H'=H$. This follows from the fact that for any $h \in \overline{H}$, $\overline{H}=hH=Hh$, so that for any $a, b \in \overline{H}$, $ab \in H$.

If the index of H in G is >2, then $H'=G$. We prove this by showing $G \subset H'$. Let g be any element of G and let a be any element of $\overline{H}$ which is not an element of gH. Such an a exists since the index of H in G exceeds 2. Then $g=ax$ has a solution $x \in \overline{H}$, since $x \in H$ implies $a \in gH$ contrary to hypothesis. In either case H' is clearly a normal subgroup.

Also solved by J. D. Baum, W. J. Blundon, G. U. Brauer, R. H. Breusch, J. R. Byrne, N. J. Fine, A. H. Frey, Jr. and R. J. Wisner, V. D. Gokhale, D. S.

Greenstein, J. Horváth, H. H. Johnson, Tung-Po Lin, D. C. B. Marsh, Katuyosi Memezawa, T. O. Moore, G. C. Preston, Azriel Rosenfeld, H. D. Ruderman, E. V. Schenkman, J. D. Thomas, J. V. Whittaker, and the Proposer. Late solution by J. B. Kelly.

Quadratic Residues

4641 [1955, 371]. *Proposed by E. P. Starke, Rutgers University*

Given N, however large, prove that there exist primes P such that each of the numbers $1, 2, 3, \cdots, N$ is a quadratic residue (mod P).

I. *Solution by Harley Flanders, University of California, Berkeley.* We shall prove, by induction on N, that there are infinitely many such primes. The proof uses (1) the Dirichlet theorem that each arithmetic progression $x \equiv a \pmod{m}$, where $(a, m) = 1$, contains an infinitude of primes, and (2) an easy consequence of the Gauss quadratic reciprocity law which asserts that if $m_1 \equiv m_2 \pmod{8ak}$, then $(a \mid m_1) = (a \mid m_2)$, where $(a \mid m)$ is the Jacobi symbol and k is arbitrary.

For $N = 1$, the result is clear. Suppose now that $(1 \mid P) = (2 \mid P) = \cdots = (N-1 \mid P) = 1$ has an infinitude of solutions, P, but that almost all of these satisfy $(N \mid P) = -1$. Take two such, P_1 and P_2. Then $(1 \mid P_1P_2) = (2 \mid P_1P_2) = \cdots = (N-1 \mid P_1P_2)$ and $(N \mid P_1P_2) = (N \mid P_1)(N \mid P_2) = 1$. Any prime Q satisfying $Q \equiv P_1P_2 \pmod{8 \cdot N!}$ will satisfy $(1 \mid Q) = \cdots = (N \mid Q) = 1$, and there are infinitely many such Q.

II. *Solution by Virginia S. Hanly, Ohio State University.* Let $p_1, p_2, \cdots, p_n$ be the odd primes $\leqq N$. The Legendre symbol $(x \mid P)$ is factorable, whence $(x \mid P) = 1$ for $x = 1, 2, \cdots, N$ if and only if $(x \mid P) = 1$ for $x = 2, p_1, p_2, \cdots, p_n$. Let S be the system $\{1 + 8p_1p_2 \cdots p_nt\}$. Let P be one of the infinitely many primes (by Dirichlet's theorem) in S. Then $P \equiv 1 \pmod{x}$ for $x = 8, p_1, p_2, \cdots, p_n$, whence $(2 \mid P) = 1$ and $(P \mid x) = 1$ for $x = p_1, p_2, \cdots, p_n$. Since $P \equiv 1 \pmod 4$ it follows from the reciprocity law that $(x \mid P) = 1$ for $x = 2, p_1, p_2, \cdots, p_n$ so that each of the numbers $1, 2, \cdots, N$ is a quadratic residue (mod P).

Also solved by W. J. Blundon, R. H. Breusch, Leonard Carlitz, J. Horváth, D. C. B. Marsh, A. Walfisz, and Morgan Ward.

Editorial Note. Walfisz points out a more general result of V. R. Fridlender, On the least nth power non-residue, *Doklady Akad. Nauk USSR (N.S.)* 66, (1942), 351–352. See also *Mathematical Reviews*, 10 (1949), 684.

A Minimax Problem

4642 [1955, 371]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn.*

Let $a_2, a_3, \cdots, a_n$ be arbitrary. Prove that there must be an x in $(0, 1)$ such that

$$| x + a_2x^2 + \cdots + a_nx^n | \geqq \frac{1}{n} \tan \frac{\pi}{4n}.$$

Solution by D. S. Greenstein, University of Pennsylvania. Let

$$P(x) = x + a_2x^2 + \cdots + a_nx^n.$$

Since for all real x

$$|P(x)| \geqq |\operatorname{Re} P(x)| = |x + (\operatorname{Re} a_2)x^2 + \cdots + (\operatorname{Re} a_n)x^n|$$

it will suffice to consider the case where $P(x)$ is a real polynomial.

Suppose

$$|P(x)| < \frac{1}{n}\tan\frac{\pi}{4n} \qquad (0 \leqq x \leqq 1).$$

Consider now the polynomial

$$P^*(x) = (-1)^n\left(\frac{1}{n}\tan\frac{\pi}{4n}\right)T_n\left\{x\left(1+\cos\frac{\pi}{2n}\right)-\cos\frac{\pi}{2n}\right\},$$

$T_n(x)$ being that polynomial of Tchebychev for which $T_n(\cos\theta)=\cos n\theta$. It is readily verified that $P^*(0)=0$ and $P^{*\prime}(0)=1$, so that $P^*(x)$ is of the same form as $P(x)$. Furthermore

$$|P^*(x)| \leqq \frac{1}{n}\tan\frac{\pi}{4n},$$

$$P^*(x_k) = (-1)^{n+k}\frac{1}{n}\tan\frac{\pi}{4n}, \qquad (k = 1, \cdots, n),$$

where $0<x_1< \cdots <x_n=1$, and

$$x_k = \frac{\cos\dfrac{\pi}{2n} - \cos\dfrac{k\pi}{n}}{1+\cos\dfrac{\pi}{2n}}.$$

It is now obvious from the graph that there must exist points ξ_k, $(k=1, \cdots, n-1)$ such that $x_k>\xi_k>x_{k+1}$ and $P^*(\xi_k)=P(\xi_k)$. Since, furthermore, $P(x)-P^*(x)$ has at least a double zero at the origin, it follows that $P(x)-P^*(x)$ has at least $n+1$ zeros, hence that $P(x)=P^*(x)$. This being impossible, there must be at least one x such that $0<x\leqq 1$ and $|P(x)|\geqq\tan(\pi/4n)/n$.

Also solved by R. H. Breusch, Gabor Szegö, Chih-yi Wang, and the Proposer.

Editorial Note. Szegö points out that the present result, and more, is implied in a paper by I. Schur, Maximum des absoluten Betrages eines Polynoms, *Math. Zeitschrift*, 4 (1919), p. 280. Wang finds an equivalent result in S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d' une Variable Réele*, Paris, 1926, pp. 36–37.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Elementary Statistics for Students of Social Science and Business. By R. Clay Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

The underlying theme of this text is well-enunciated by the author in his preface in the following four statements:

"1. Statistics is the science of making decisions based upon incomplete knowledge.

2. Implicit in making a decision is the acceptance of one of two alternative hypotheses.

3. Associated with the acceptance of either hypothesis is the risk that the one selected is incorrect.

4. The function of statistics is to aid in the formulation of the alternative hypotheses and the evaluation of the associated risks of being wrong."

In this reviewer's opinion, the author is unusually successful in maintaining this principle as the central theme of his text. Sprowls' repeated insistence on the formulation of rules of inductive behavior and the evaluation of their consequences by means of operating characteristics as the *raison d'être* of statistics is convincingly and effectively made a part of the student's belief.

Two sampling experiments introduced in the first chapter form a basis for giving the student concrete experience with the problems of inference and are used also in later sections of the text to illustrate further developments. In Chapter 3 the student is faced with the problem of applying these concepts to a personal decision problem, whose value, to quote the author, "lies in its ability to bring home to the student the basic methods of statistics and their relationship to everyday decisions, even though such decisions may not require the use of statistical data." A small research project, suggested to be started at about the middle of the course, is well-presented in the discussion of Chapter 6, "Statistical Research Methodology", which can be read profitably by the student without benefit of lectures. Again, the author's comment, "Students thus realize the complexity of apparently simple problems", seems significant.

In general, the problems throughout the text are meaningful and varied in fields of application. Except for the four chapters on Time Series and Index Numbers, the text material and exercises are in no way restricted to the business and economic aspects of statistics.

Among more specific highlights of the text are:

1. The author's use of pre-learning and anticipation. Concepts are frequently introduced briefly in one context, then later presented more formally and developed in detail.

2. The requirement of a certain amount of abstract reasoning imposed by the text. This reviewer shares the author's belief that there has been too much writing down to the level of the student. In Chapter 2, after a brief discussion of classical notions of probability, the author presents quite simply an axiomatic definition of discrete probability. (A later brief presentation of the continuous case seems less fortuitous.)

3. This is very definitely a text in the category of the so-called "non-mathematical" statistics treatments; in general, no proofs or derivations are given, but the author is usually scrupulous in presenting underlying assumptions and in making precise statements of what are essentially mathematical theorems to be employed. It is because of the author's success in this last respect, in the setting of the central theme of the text cited earlier, that this treatment avoids the frequently encountered recipe-book stigma. To this reviewer it appears that more "mathematical reasoning" has been incorporated into Sprowls' text than in other texts of this category.

An exception of the statement concerning the author's scruples in presenting underlying assumptions seems to be present in his discussion of optimal properties of various tests. It is doubtless good pedagogy to rely on the student's intuition at the outset in picking, for example, a left-hand tail region for rejecting the hypothesis $p = p_0$ in favor of the alternative hypothesis $p = p_1$, with $p_1 < p_0$, for the class of binomial distributions. It would seem more in the spirit of Sprowls' philosophy, however, then to make some statement other than the dogmatic assertion (not as a theorem, but with the impact of "obviously") that such a choice in this case leads to a test with best power. (A reference is given at the end of Chapter 4 to Neyman's text.)

Finally, the author's selection of topics seems to have been made judiciously and realistically. For students in fields other than business or economics who may not be interested in the chapters on Time Series and Index Numbers, material only briefly introduced in the last chapter would furnish a springboard for study of topics such as inference problems involving two or more samples, which otherwise would have to be omitted.

The treatment of a time series from the point of view of a stochastic process, with statistical tests suggested for the detection of systematic components, is skillfully presented and, again, is consistent with the four-fold theme of the text. The student of economics and business is initiated in these chapters into some of the current controversies in these fields.

An instructor's manual furnished by the author gives useful suggestions based on his experience in teaching the course, including a suggested time-schedule based on a semester course of 45 lectures and 15 two-hour laboratory periods. The manual is also illuminating in conveying Sprowls' philosophy and enthusiasm with some prospect of contagion.

GRACE E. BATES
Mt. Holyoke College

Vector Analysis, By H. E. Newell, Jr. McGraw-Hill Book Company, Inc., New York, 1955. 216 pp. $5.50.

This new book by a well-known physicist at the Naval Research Laboratory is intended for use in a one-semester course either at the undergraduate or beginning graduate level. Part I of *Vector Analysis* treats the algebra and calculus of vectors and includes a chapter reviewing some concepts from the calculus. Part II, which occupies somewhat less than half of the book, deals with applications of the theory to mechanics and electromagnetism.

Dr. Newell, in the preface to his book, states that he is writing primarily for the engineer and physicist and hopes that his readers will make vector concepts an integral part of their physical and intuitive reasoning. He leans heavily on the use of problems to complete and enrich the text material, which is, perhaps, a little sparse for general use. Indeed, a general criticism of the book is that no freedom of choice has been left to the teacher in the selection of topics to be studied. Neither would this book be of too much value on a reference shelf, for many of the most useful results are only to be found scattered in the exercises.

The author has been quite careful in his statements of theorems and definitions, but has omitted most of the arguments necessary to make his proofs rigorous, leaving these to be filled in, as exercises, by his readers. He has, however, made the arguments plausible and has, in some cases, indicated how a rigorous proof would be carried out.

The format of the book is neat and attractive, but the figures, in many instances, could be improved visually.

S. C. Lowell
New York University

CORRECTION

My review of Vance's *Trigonometry** (Addison-Wesley, 1954) erroneously claimed that this book contains "the first completely satisfactory derivation of composite angle formulas to be found in an elementary text." I should have added the qualifying phrase, "to the reviewer's knowledge", to that sentence. The book *Elementary Analysis*, by K. O. May, among others, contains a somewhat similar derivation. I hereby apologize to Professor May and all concerned.

J. G. Hocking
Michigan State College

* This Monthly, vol. 62, 1955, pp. 458–459.

NEW BOOKS RECEIVED

How to Calculate Quickly—Rapid Methods in Basic Mathematics. By Henry Sticker. New York, Dover Publications, Inc., 1956. 256 pages. Paperbound $1.00.

Risk and Gambling, The Study of Subjective Probability. By John Cohen and Mark Hansel. New York, Philosophical Library, Inc., x+153 pages. $3.50.

Solid Geometry. By D. T. Sigley and W. T. Stratton. New York, The Dryden Press, Inc., 1956. x+197 pages. $2.60.

Integral Functions. No. 44 in Cambridge Tracts in Mathematics and Mathematical Physics. General Editors P. Hall and F. Smithies. By M. L. Cartwright. New York, Cambridge University Press, 1956. 135 pages. $3.50.

Elementare Differentialgeometrie. By W. Haack. Basel, Verlage Birkhäuser, 1955. viii+239 pages. SFr. 22.

Combinatorial Topology, Vol. I. By P. S. Aleksandrov. Rochester, New York, The Graylock Press, 1956. xvi+225 pages. $4.95.

Weather Analysis and Forecasting, Vol. I, Motion and Motion Systems, 2nd Ed. By Sverre Petterssen. New York, McGraw-Hill Book Co., 1956. xix+428 pages. $8.50.

Calculus. By Jack R. Britton. New York, Rinehart and Company, Inc., 1956. xiv+584 pages. $6.50.

Royal Society Mathematical Tables, 2, Rectangular-Polar Conversion Tables. By E. H. Neville. New York, Cambridge University Press, 1956. xxxii+109 pages. $5.50.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

LECTURES IN MATHEMATICAL STATISTICS AT CATHOLIC UNIVERSITY OF AMERICA

A lecture series in the field of mathematical statistics will be held by the Department of Mathematics of Catholic University at the beginning of the academic year 1956–57, probably in October. It is planned to invite prominent mathematical statisticians from the east coast to give addresses. The cooperation of other universities and interested organizations in the area will be sought. Enquiries concerning the lecture series may be addressed to the Department of Mathematics, Catholic University of America, Washington, D. C.

MATHEMATICAL MONOGRAPH NO. 16

A university library is interested in purchasing a copy of R. D. Carmichael, "Diophantine Analysis," published by Wiley in 1915 as Mathematical Monograph No. 16. Anyone having a copy for sale should write to the office of the Mathematical Association of America.

WILLIAM B. ORANGE MATHEMATICS PRIZE COMPETITION

The Mathematics Department of Los Angeles City College held the sixth annual William B. Orange Mathematics Prize Competition for students in Los Angeles High Schools on May 18, 1956. There were 167 students enrolled in the Competition. These students represented 35 local high schools.

Team winner was Van Nuys High School. Individual winners were Michael Raugh of Van Nuys High School, Howard Weisberg and Harold Stark of Hamilton High School, Louis Jaeckel of Van Nuys High School, and Floris Y. Tsang of Los Angeles High School.

PERSONAL ITEMS

Dean W. L. Duren, Jr., University of Virginia, will represent the Association at the inauguration of Chancellor G. C. Simpson of Mary Washington College of the University of Virginia on October 18–19, 1956.

Professor H. M. Gehman of the University of Buffalo was the delegate of the Association at the Convocation commemorating the Centennial of the Establishment of Niagara University on May 10, 1956.

The John Simon Guggenheim Memorial Foundation has announced the granting of Fellowship Awards in Mathematics and Statistics to the following for research as indicated: Professor C. A. Truesdell, III, of Indiana University, studies of the history of the mathematical theory of elasticity; Professor E. E. Moise of the University of Michigan, studies of classical problems in topology of higher dimensional euclidean manifolds; Professor J. L. Hodges, Jr., University of California, studies in the use of combinatorial analysis in mathematical statistics; Professor W. H. Fuchs of Cornell University, study of the deficient values of meromorphic functions of finite order; Professor E. W. Barankin, University of California, quantitative studies toward a theory of human behavior.

The National Science Foundation has announced the appointment of Professor E. J. McShane of the University of Virginia and Dr. Warren Weaver of the Rockefeller Foundation to the National Science Board.

Professor A. A. Albert of the University of Chicago has been appointed to membership on the Postdoctoral Fellowship Board of the National Academy of Sciences.

Professor R. P. Boas, Jr., of Northwestern University has received a two-year grant for research on trigonometric series from the National Science Foundation.

Professor Louis Brand of the University of Cincinnati has been awarded an honorary degree of Doctor of Science by the University.

Professor T. Y. Thomas of Indiana University has been promoted to Distinguished Service Professor, the University's highest academic rank.

Auckland University College, University of New Zealand, announces: Professor H. G. Forder, who has resigned as Chairman of the Department of Mathematics, has been appointed Professor Emeritus and is continuing to teach at the College; Professor Frederick Chong of Sydney University, Australia, has been appointed Chairman of the Department.

Brown University reports the following: Dr. D. A. Buchsbaum of the University of Chicago has been appointed to an assistant professorship; Dr. W. D. Barcus of Princeton University has been appointed to an instructorship; Assistant Professor M. R. Demers of the University of Nevada has been appointed Research Assistant; Dr. Jean-Pierre Meyer has been appointed a research associate; Dr. David Harrison of Princeton University has been appointed to a teaching internship; Professor Maurice Heins and Assistant Professor John Wermer are on leave of absence for 1956–57 and are at the Institute for Advanced Study.

At Cornell University: Associate Professor Harry Pollard has been promoted to a professorship; Dr. H. D. Block, Dr. C. S. Herz, and Dr. D. B. Ray have been promoted to assistant professorships; Dr. Isaac Namioka of the University of California and Dr. Wolfgang Rindler of the University of London have been appointed to instructorships; Dr. Jerome Blackman of Syracuse University has been appointed Acting Assistant Professor; Dr. Ambikeshwar Sharma of the University of Lucknow has been appointed Visiting Assistant Professor; Professor G. A. Hunt will be on leave at Princeton University during 1956–57; Dr. L. A. Rubel has been awarded a National Science Foundation Fellowship and will spend the year 1956–57 at the Institute for Advanced Study.

Emory University announces the following: Professor C. G. Latimer has retired as Chairman of the Department of Mathematics but will continue as Professor of Mathematics; Associate Professor Trevor Evans has been appointed Chairman of the Department; Mr. E. E. Grace of the University of North Carolina and Assistant Professor Henry Sharp of Georgia Institute of Technology have been appointed to assistant professorships; Mr. F. L. Hardy, formerly a graduate assistant at the University, has been appointed to an instructorship.

Illinois Institute of Technology reports: Mr. R. J. Mihalek and Mr. Barth Pollak, a fellow at Princeton University, have been appointed to instructorships; Associate Professor S. F. Bibb has retired.

Indiana University announces: Professor J. W. T. Youngs has been appointed Chairman of the Department of Mathematics and Director of the Graduate Institute for Mathematics and Mechanics; the Institute will become that portion of the Mathematics Department concerned with graduate affairs.

At Massachusetts Institute of Technology: Dr. Hartley Rogers, Jr., formerly a Visiting Lecturer at the Institute, and Assistant Professor I. M. Singer of the University of California at Los Angeles, have been appointed to assistant professorships; Dr. W. L. Baily of Princeton University, Dr. R. T. Prosser of Duke University, and Dr. E. M. Stein, University of Chicago, have been appointed C. L. E. Moore Instructors; Mr. K. J. Hoffman of the University of California at Los Angeles and Mr. Leonard Roberts, previously a research associate at the Institute, have been appointed to instructorships; Dr. M. A. Kervaire of Berne University, Switzerland, has been appointed Visiting Lecturer in Mathematics; Professor J. B. Diaz of the University of Maryland and Dr. S. M. Ulam, senior scientist at Los Alamos Scientific Laboratory, have been appointed to visiting professorships; Assistant Professor R. A. Clark of Case Institute of Technology has been appointed Visiting Assistant Professor; Dr. K. F. Roth, lecturer at the University of London, and Dr. J. T. Stuart of the National Physical Laboratory, Teddington, England, have been appointed to visiting lectureships; Dr. Marcel Berger of the Institute of Mathematics, University of Strasbourg, has been appointed Research Associate; Associate Professor R. V. Kadison of Columbia University is on partial leave and is also Research Associate at the Institute; Assistant Professor J. F. Nash, Jr. is on leave of absence at the Institute for Advanced Study; Associate Professor G. B. Thomas, Jr., is on leave of absence at Stanford University; Professors G. W. Whitehead and Norbert Wiener have returned from their leaves of absence.

Northwestern University reports the following: Dr. W. C. Fox of Massachusetts Institute of Technology has been appointed to an assistant professorship; Mr. W. L. Hoyt of the University of Chicago has been appointed to an instructorship; Assistant Professor J. A. Zilber of the University of Illinois has been appointed Lecturer.

Purdue University announces the following: Professor C. F. Kossack, previously director of the Statistical Laboratory, has been appointed Head of the Department of Mathematics; Professor V. L. Anderson is now Head of the Statistical Laboratory; Dr. Paul Brock of the ElectroData Corporation, Pasadena, California, has been appointed to an associate professorship; Assistant Professors V. L. Anderson, John Dyer-Bennet, Leonard Gillman, Meyer Jerison, and Henry Teicher have been promoted to associate professorships; Dr. A. H. Copeland, Jr., Dr. G. L. Krabbe, Mr. Clarence Lane, and Dr. C. J. Neugebauer have been promoted to assistant professorships; Mr. W. A. Stock who is at the Hammond Extension Center has been promoted to an assistant professorship; Professor Melvin Henriksen is on leave of absence during 1956–57 and is at the Institute for Advanced Study on an Alfred P. Sloan Foundation grant; Dr. A. H. Copeland received a Purdue University research grant for the summer of 1956.

United States Naval Academy reports: Associate Professors R. P. Bailey, T. J. Benac, L. H. Chambers, and J. P. Hoyt have been promoted to professorships;

Assistant Professor Joseph Milkman has been promoted to an associate professorship; Associate Professor R. W. Rector is on leave of absence for 1956–57 at the Ramo-Wooldridge Corporation, Los Angeles, California.

At the University of Nebraska: Dr. D. L. Guy of Washington University and Miss F. Eloise Pool have been appointed to instructorships; Associate Professor W. G. Leavitt has been promoted to a professorship; Dr. F. W. Anderson has been promoted to an assistant professorship; Professors F. W. Anderson and M. A. Basoco received Faculty Summer Research Fellowships; Professor H. B. Ribeiro is on leave of absence for six months at the University of Münster.

Miss Lucille J. Albers, previously a data processing mathematician for Land-Air, Dayton, Ohio, has a position as a mathematician at the Air Force Armament Center, Eglin Air Force Base, Florida.

Assistant Professor E. L. Arnoff of Case Institute of Technology has been promoted to an associate professorship. Professor Arnoff is also Assistant Director of the Operations Research Group.

Dr. Archie Blake, formerly an advisory engineer for Westinghouse Electric Corporation, Baltimore, Maryland, has a position as a systems staff mathematician at the Bendix Aviation Corporation, Detroit, Michigan.

Mr. R. M. Blumenthal of Cornell University has been appointed to an instructorship at the University of Washington.

Mr. A. P. Boblétt, formerly chief statistician at the U. S. Naval Ammunition Depot, Crane, Indiana, has accepted a position as Mathematical Consultant at the U. S. Naval Ordnance Laboratory, Corona, California.

Dr. S. R. Bodner, recently a senior mathematician at Republic Aviation Corporation, Farmingdale, New York, has a position as a senior scientist at the Avco Manufacturing Company, Stratford, Connecticut.

Mr. L. M. Bostick, previously a graduate assistant at Eastern New Mexico University, has a position as a nuclear engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Professor Louis Brand, head of the Department of Mathematics of the University of Cincinnati, has been appointed Whitney Visiting Professor at Trinity College, Hartford, Connecticut, for the academic year 1956–57.

Professor W. B. Carver of Cornell University was in charge of the program in mathematics for the Shell Merit Fellows at the University during the summer of 1956. This program was sponsored by the Shell Oil Company for high school science and mathematics teachers. Dr. H. D. Block of the university assisted Professor Carver in this program.

Dr. C. C. Chang of Cornell University has been appointed to an assistant professorship at the University of Southern California.

Acting Dean J. O. Chellevold of Wartburg College has been appointed Dean.

Assistant Professor S. D. Conte of Wayne University is on sabbatical leave and is with the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor E. A. Davis of the University of Nevada has been ap-

pointed to an assistant professorship at the University of Utah.

Assistant Professor R. B. Davis of the University of New Hampshire has been appointed to an associate professorship at Syracuse University.

Associate Professor R. D. Depew of State Teachers College, Florence, Alabama, has a position as a mathematician and instructor at Lincoln Laboratory, Lexington, Massachusetts, for the International Business Machines Corporation.

Assistant Professor H. W. Doss, Jr., of the University of Omaha has been appointed a research associate at the Engineering Research Institute, University of Michigan.

Mr. D. G. B. Edelen of Johns Hopkins University has accepted a position as an engineer for the Glenn L. Martin Company, Middle River, Maryland.

Assistant Professor M. P. Emerson of Harpur College has been appointed to an associate professorship at Southwest Missouri State College.

Assistant Professor J. M. Feld of Queens College has been promoted to an associate professorship.

Assistant Professor W. R. Ferrante of Lafayette College has been appointed to an assistant professorship in the Department of Mechanical Engineering at the University of Rhode Island.

Assistant Professor R. I. Fields of the Speed Scientific School, University of Louisville, has been granted a year's leave of absence by the University and is serving as an assistant professor for the year at Virginia Polytechnic Institute.

Associate Professor D. T. Finkbeiner, II, of Kenyon College has been promoted to a professorship.

Dr. H. K. Flesch of Cornell University has accepted a position with the Federal Telecommunication Laboratories, Nutley, New Jersey.

Associate Professor Harriet M. Griffin of Brooklyn College has been promoted to a professorship.

Professor D. W. Hall of the University of Maryland has been appointed to a professorship at Harpur College.

Associate Professor P. R. Halmos of the University of Chicago has been promoted to a professorship.

Associate Professor A. E. Halteman, previously of the University of Idaho, has a position with the Electronic Defense Laboratory, Mountain View, California.

Dr. Carl Hammer, formerly a senior research engineer at Franklin Institute, is Director of the UNIVAC European Computer Center, Germany.

Associate Professor E. H. C. Hildebrandt of Northwestern University has been awarded a Fulbright lectureship to teach mathematics at the Higher Teachers Training College in Baghdad, Iraq, during the academic year 1956–57.

Associate Professor P. G. Hodge, Jr., of Polytechnic Institute of Brooklyn has been promoted to the position of Professor of Applied Mechanics.

Mr. J. G. Horne, Jr., of Tulane University has been appointed to an assistant professorship at the University of Kentucky.

Dr. Rufus Isaacs of Lockheed Aircraft Corporation, Los Angeles, California, has accepted a position with the Hughes Aircraft Company, Culver City, California.

Mr. D. H. Jones, previously a teaching fellow at the University of Washington, is employed as a scientist at the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Associate Professor C. W. Jordan, Jr., of Williams College has been promoted to a professorship.

Associate Professor Irving Kaplansky of the University of Chicago has been promoted to a professorship.

Assistant Professor L. A. Kokoris of Washington University has been appointed Visiting Lecturer at Yale University.

Mr. S. R. Lenihan, formerly a thermodynamics engineer for Consolidated-Vultee Aircraft Corporation, San Diego, California, has a position as a nuclear engineer for Ralph M. Parsons Company, Los Angeles, California.

Mr. A. M. Linn has a position as a mathematician with Pratt-Whitney Aircraft Corporation, East Hartford, Connecticut.

Dr. Mark Lotkin, previously a mathematician at Republic Aviation Corporation, Farmingdale, New York, is Chief of the Mathematical Section of the Advanced Development Division, Avco Manufacturing Company, Stratford, Connecticut.

Assistant Professor Nathaniel Macon of Alabama Polytechnic Institute is employed now as a numerical methods analyst with the General Electric Company, Cincinnati, Ohio.

Dr. W. E. Mientka of the University of Massachusetts has been appointed to an assistant professorship at the University of Nevada.

Dr. Clair E. Miller of the Ramo-Wooldridge Corporation has accepted a position as research mathematician with the California Research Corporation, Richmond, California.

Dr. M. E. Muller of Cornell University has accepted a position with International Business Machines Corporation, New York, New York.

Dr. Rufus Oldenburger, director of research for the Woodward Governor Company, Rockford, Illinois, has been appointed Professor of Engineering Sciences and Mechanical Engineering at Purdue University.

Mr. F. J. Palas, formerly a graduate student at the University of Oklahoma, has been appointed to an assistant professorship at Southern Methodist University.

Professor T. S. Peterson of Portland State College has been appointed Chairman of the Division of Science.

Assistant Professor Gustave Rabson of Antioch College has been promoted to an associate professorship.

Mr. Alfred Reichenthal, previously an engineer with the American Bosch Arma Corporation, New York, New York, has accepted a position as a research engineer with Hughes Aircraft Company, Los Angeles, California.

Mr. A. T. Rice, formerly a mathematician for the United States Air Force, Washington, D. C., is now a test calculations liaison specialist for the General Electric Company, Lynn, Massachusetts.

Associate Professor C. L. Rich of Stout State College is retiring from teaching this year.

Dr. P. R. Rider of the Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, spent the summer on temporary duty with the Summer Research Group of the Holloman Air Development Center, New Mexico.

Dr. Lawrence Rosenfeld, previously a senior engineer at Raytheon Manufacturing Company, has a position as Head of Operations Research and Mathematical Services, Melpar, Inc., Cambridge, Massachusetts.

Assistant Professor W. C. Royster of Alabama Polytechnic Institute has been appointed to an assistant professorship at the University of Kentucky.

Mr. Jerome Sacks of Cornell University has been appointed to an instructorship at California Institute of Technology.

Dr. O. P. Sanders of Oklahoma Agricultural and Mechanical College has been appointed to an associate professorship at Southeastern State College, Oklahoma.

Dr. R. C. Seber of the State University of Iowa has been appointed to an assistant professorship at Western Michigan College of Education.

Assistant Professor A. L. Shields of Tulane University has been appointed to a visiting assistant professorship at the University of Michigan.

Associate Professor James Singer of Brooklyn College has been promoted to a professorship.

Associate Professor Jerome C. Smith of Lafayette College has been appointed Professor and Head of the Department of Mathematics of High Point College.

Mr. C. B. Solloway of the California Institute of Technology has been appointed to an assistant professorship at the University of Southern California.

Dr. W. C. Swift of Cornell University has accepted a position with Bell Telephone Laboratories, Murray Hill, New Jersey.

Mr. P. Y. Tani, previously a research engineer at North American Aviation, Los Angeles, California, has been appointed a research associate at Willow Run Laboratories, University of Michigan, Ypsilanti, Michigan.

Mr. R. P. Tapscott, a research assistant at Purdue University, has a position as a project engineer at the Allison Division of General Motors, Indianapolis, Indiana.

Mr. R. L. Uschold of Nazareth College, Rochester, New York, has been appointed to an instructorship at Canisius College.

Dr. M. A. Woodbury, previously at George Washington University, has been appointed Research Professor of Mathematics at New York University.

Dr. C. T. Yang, a member of the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Pennsylvania.

Dr. R. E. Zink has been appointed to an assistant professorship at Purdue University.

Professor Emeritus W. J. Berry of Polytechnic Institute of Brooklyn died on March 5, 1956. He was a charter member of the Association.

Professor Emeritus A. D. Butterfield of the University of Vermont died on August 22, 1955. He was a member of the Association for 31 years.

Professor G. R. Clements (retired) of the United States Naval Academy died on March 25, 1956. He was a member of the Association for 39 years and was a Governor of the Maryland-District of Columbia-Virginia Section from 1950–53.

Professor V. D. Gokhale, Chairman of the Department of Mathematics of Atlanta University, died on June 1, 1956.

Professor Emeritus G. W. Mullins of Columbia University died on March 11, 1956. He was a member of the Association for 36 years.

Captain A. R. Park, professor of mathematics and director of guidance and testing at Wentworth Military Academy, died on March 23, 1956. A memorial medal award (known as Capt. A. R. Park Memorial Medal for Proficiency in Mathematics) has been established and will be given to the junior college cadet excelling in mathematics.

Dr. M. M. Slotnick, a geophysical consultant with the Standard-Vacuum Oil Company, New York, New York, died May 7, 1956. He was a member of the Association for 19 years.

Professor Emeritus P. F. Smith of Yale University died June 3, 1956.

Associate Professor C. F. Strobel of North Carolina State College died April 21, 1956. He was a member of the Association for 20 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1956, by a mail vote of the membership of the Association in the Sections indicated:

Illinois	E. C. Kiefer, Millikin University
Iowa	Bernard Vinograde, Iowa State College
Louisiana-Mississippi	Z. L. Loflin, Southwestern Louisiana Institute
Maryland-District of Columbia-Virginia	O. J. Ramler, Catholic University of America

Michigan	B. M. Stewart, Michigan State University
Minnesota	G. K. Kalisch, University of Minnesota
Philadelphia	N. J. Fine, University of Pennsylvania
Southern California	P. H. Daus, University of California at Los Angeles
Texas	C. R. Sherer, Texas Christian University

Almost 35% of those eligible to vote in the above elections cast ballots for the election of a sectional governor. In the case of two sections, votes were received from more than 50% of the membership. In the Louisiana-Mississippi Section, the percentage of votes cast was 57% and in the Iowa Section 52%.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 127 persons have been elected to membership by the Board of Governors on applications duly certified.

S. L. ANDERSON, Student, Seattle University.

G. A. ANDREWES, M.S. (Massachusetts) Instr., Technical High School, Springfield, Mass.

BROTHER AUBIN, C.F.X., M.S. (St. John's U.) Chairman, Department of Mathematics, Mount St. Joseph High School, Baltimore, Md.

P. M. BAILYN, Student, Cooper Union.

J. G. BENNETT, B.S. (Trinity U.) Champaign, Illinois.

R. L. BERGGREN, M.A. (Minnesota) Computing Engr., Douglas Aircraft Co., Long Beach, Calif.

W. S. BISHOP, M.S. (Arkansas) Asst. Professor, Howard College.

R. E. BLEWSTER, JR., Student, Agricultural and Mechanical College of Texas.

CAROL A. A. BLOOMQUIST, Student, University of Minnesota.

N. R. BONAM, Student, Kansas State College.

LILLIAN K. BRADLEY, M.A. (Michigan) Chairman, Department of Mathematics, Texas College.

G. Y. BREITBARD, Student, California Institute of Technology.

W. J. BUCKINGHAM, Student, Stanford University.

R. E. BURGESS, B.S. (Purdue) Senior Math., Librascope, Burbank, Calif.

W. H. BURGIN, JR., Student, Dartmouth College.

W. R. CARNES, M.S. (Georgia I.T.) Instr., Georgia Institute of Technology.

BOMSHIK CHANG, M.A. (Seoul National U.) Teaching Asst., University of British Columbia.

R. K. CLARK, Student, Texas Christian University.

E. L. COHEN, Student, Harvard University.

R. T. CRAIG, Student, College of the Holy Cross.

P. L. CRAWLEY, Student, California Institute of Technology.

D. M. DAHM, Student, Rice Institute.

VASSILI DAIEV, Sea Cliff, New York.

E. A. DAVIS, Ph.D. (California) Asst. Professor, University of Utah.

D. A. DAWSON, Student, McGill University.

E. I. DEATON, B.A. (Hardin-Simmons) Teaching Asst., University of Texas.

R. S. DINSMORE, Student, Princeton University.

MRS. NANCY M. DISMUKE, M.S. (Brown) Math., Oak Ridge National Lab.

M. H. DIXON, Student, Siena College.

A. C. DOWNING, JR., M.S. (Michigan) Senior Math., Oak Ridge National Lab.

RUDOLPH DOWNS, Student, Rutgers University.

UNDERWOOD DUDLEY, Student, Carnegie Institute of Technology.

MANNING FEINLEIB, Student, Cornell University.

T. S. FERGUSON, Ph.D. (California) Lecturer, Statistics, University of California.

W. H. FLEMING, Ph.D. (Wisconsin) Asst. Professor, Purdue University.

D. A. FORD, Student, Occidental College.

J. W. FOTHERGILL, JR., B.A. (Sacramento S.C.) Math., University of California Radiation Lab.

LT. D. A. FRANKS, M.S. (Howard) Chief, Internal Guidance Section, Ordnance Guided Missiles School, Redstone Arsenal, Huntsville, Ala.

W. Y. GATELEY, S.M. (M.I.T.) Instr., Clarkson College of Technology.

L. D. GOLDBERG, Student, Yale University.

MRS. ELNA H. GRAHN, M.S. (Wisconsin) Visiting Instr., University of Idaho.

H. S. GREENFIELD, M.S. (Brigham Young) Res. Physicist, Dugway Proving Ground, Utah.

R. K. GRUENEWALD, M.A. in Ed. (Washington) Teacher, Eliot School, St. Louis, Mo.; Instr., Washington University.

MARTHA J. GUTHRIE, Student, Arizona State College.

IRWIN GUTTMAN, Ph.D. (Toronto) Asst. Professor, University of Alberta.

H. N. HADLEY, B.A. (Whitman) Head, Statistical Division, Inspection and Test Department, Naval Powder Factory, Indian Head, Md.

P. R. HARNESS, Student, Tennessee Polytechnic Institute.

R. W. HARRUFF, Student, Kent State University.

BERT HENRY, B.S. (Western Reserve) Director, Schillinger Center, Cleveland, Ohio.

PAUL HERZBERG, Student, Queen's University.

A. L. HESS, Ph.D. (Wisconsin) Asso. Professor, Montana State College.

F. H. HILDEBRAND, Student, Kent State University.

S. K. HOWRY, M.A. (California) Math., Remington Rand Univac, St. Paul, Minn.

R. C. HUBER, B.A. (Lehigh) Math. Engr., Renner Industrial Consultants, Philadelphia, Pa.

MRS. LOUISE F. HUTCHINSON, M.A. (Bryn Mawr) Grad. Student, Bryn Mawr College.

J. F. JAKOBSEN, A.M. (Missouri) Instr., University of Missouri.

N. M. KENDALL, Student, Baylor University.

P. L. KINGSTON, B.A. (Colgate) Grad. Student, University of Rochester.

P. E. KLEBE, JR., M.A. (Yale) Instr., Brooklyn Polytechnic Institute.

BARBARA F. KOROTKIN, Student, Temple University.

B. F. KRAMER, Student, City College of the City of New York.

H. F. KREIMER, JR., Student, Yale University.

THELMA E. KROPPER, Student, Skidmore College.

J. G. LEGHORN, M.S. (Wisconsin) Instr., University of Colorado.

D. A. LEWINSKI, M.S. (Manitoba) Member, Technical Staff, Bell Telephone Labs., New York, N. Y.

THERESA M. LOTITO, Student, LeMoyne College.

R. A. MALLGREN, Student, Wayne University.

K. O. MALMQUIST, JR., A.A. (Modesto Jr.C.) Math., University of California Radiation Lab.

WALLACE MANHEIMER, M.A. (Columbia) Chairman, Department of Mathematics, Franklin K. Lane High School, Flushing, N. Y.

ZACHARY MARTIN, Student, California Institute of Technology.

R. H. MCCOBB, Ed.D. (Temple) Asso. Professor, New Jersey State Teachers College, Glassboro.

B. J. MCDONALD, B.A. (Northland) Physicist, U. S. Navy Mine Defense Lab., Panama City, Fla.

R. A. MELTER, Student, Cornell University.

G. W. MELTON, JR., Student, Knox College.

FLORENCE M. MILLER, A.M. (Loyola) Math. Teacher, Wright Junior College.

H. G. MOORE, B.S. (Utah) Grad. Asst., University of Utah.

KENNETH MOOSMAN, Student, Utah State Agricultural College.

D. D. MORRISON, M.A. (Wayne) Res. Asso., Wayne University.

M. E. MULLER, Ph.D. (U.C.L.A.) Instr., Cornell University.

AMIN MUWAFI, B.S.E. (Florida) Grad. Asst., University of Florida.

D. E. NAGLE, Student, Arizona State College.

W. A. NASH, Ph.D. (Michigan) Asso. Professor, Engr. Mech., University of Florida.
D. A. NELSON, Student, Knox College.
RICHARD NUSINOW, B.S. (Chicago) Instr., Illinois Institute of Technology.
E. H. OBERST, Student, John Carroll University.
REV. D. J. O'BRIEN, S.J., M.A. (Bellarmine) Grad. Student, Fordham University.
W. G. OELLERICH, B.S. (Georgia) Teacher, University of Georgia, Augusta.
F. C. OGG, Ph.D. (Illinois) Chairman, Department of Mathematics, Bowling Green State University.
F. J. OSSIANDER, Student, University of Washington; Aeronautical Engr., Boeing Aircraft Co., Seattle, Wash.
H. A. PETERSON, M.A. (N.Y.U.) Math. Instr., Darien High School, Conn.
R. T. PFEIFER, Student, John Carroll University.
W. L. POPE, Student, Utah State Agricultural College.
T. A. PORSCHING, Student, Carnegie Institute of Technology.
C. W. PRICE, Consultant in Physics, Renner Industrial Consultants, Philadelphia, Pa.
L. B. RATCLIFF, M.S. (Florida S.U.) Ballistic Research Lab., Aberdeen Proving Ground, Md.
H. A. REGUSTERS, Engr. Writer, Renner Industrial Consultants, Philadelphia, Pa.
ARLINE S. REISS, Student, Brooklyn College.
MILTON ROSENBERG, Student, City College of the City of New York.
K. A. ROSS, Student, University of Utah.
F. M. ROURKE, Student, Siena College.
WALTER RUDIN, Ph.D. (Duke) Asso. Professor, University of Rochester.
M. T. SALHAB, Student, University of Texas.
D. J. SCHAEFER, Student, San Jose State College.
M. J. SCHRADER, B.S. in Ed. (Southern Ill.U.) Grad. Asst., Southern Illinois University.
W. L. SCHRADER, Student, Memphis State College.
A. H. SCOTT, A.B. (Middlebury) Grad. Asst., University of Vermont.
J. G. SEEBOLD, JR., Student, Stanford University.
S. L. SEGAL, Student, Wesleyan University.
JOAN M. SHAPIRO, Student, Wesleyan College.
R. L. SHAW, M.A. (Denver) Grad. Student, University of California.
D. J. SINE, B.S. (Frostburg S.T.C.) Grad. Student, West Virginia University.
J. L. SNELL, Ph.D. (Illinois) Asst. Professor, Dartmouth College.
A. P. STOCK, B.S. (Drexel I.T.) Electronic Engr., Renner Industrial Consultants, Philadelphia, Pa.
N. O. STOCKMAN, B.S. (Xavier U.) Aeronautical Res. Scientist, National Advisory Committee on Aeronautics, Cleveland, Ohio.
MRS. MARY R. STRANDTMANN, M.A. (Texas Tech. C.) Instr., Texas Technological College.
MRS. SALY R. STRUIK, Dr.Math. (Prague) Belmont, Massachusetts.
ARTHUR SULLIVAN, Student, Rutgers University.
C. C. THOMPSON, Student, Oklahoma Agricultural and Mechanical College.
E. R. VANCE, JR., Student, University of Redlands.
MARC VENNE, Student, University of Montreal.
D. W. WALL, Ph.D. (Michigan) Asst. Professor, University of North Carolina.
D. E. WALSTON, B.A. (A. & M.C. of Texas) Teaching Asst., University of Texas.
M. W. WEAVER, Ph.D. (Texas) Instr., University of Texas.
B. H. WESTFALL, B.A. (Kentucky) Instr., Berea College.
R. A. WIEDERANDERS, M.A. (Minnesota) Instr., Wartburg College.
C. T. YOUNG, M.A. (Vermont) Grad. Asst., University of Vermont.
HENRY ZEMEL, Student, McGill University.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 24, 1956, at Ann Arbor, Michigan, in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor C. C. Richtmeyer of Central Michigan College of Education

presided at both morning and afternoon meetings and at the luncheon and business meeting.

A total of 90 persons attended the meeting including 55 members of the Association.

The nominating committee consisting of Professors J. S. Frame, chairman, C. H. Butler, and L. E. Mehlenbacher, proposed Professor R. S. Pate of Eastern Michigan College as Chairman and Professor Fred Beeler of Western Michigan College as Secretary-Treasurer. The slate was elected unanimously.

A resolution was adopted unanimously opposing the provision of the proposed revision of the certification code for the teachers of Michigan. A copy of the resolution is to be sent to each member of the State Board of Education.

The following papers were presented at the morning and afternoon meetings:

1. *Numerical solution methods in partial differential equations using high-speed digital computers*, by Professor J. W. Carr, III, University of Michigan.

A summary is given of the developing theory of numerical solutions of partial differential equations using high-speed computers. In particular, the differences between attacks or initial- and boundary-value problems are noted. The problems of convergence of the difference equation solution to the actual solution, and of the stability of the numerical solution under round-off are also noted. Finally, some estimates are made of time necessary for the solution of certain three-dimensional problems using the fastest machines available and the fastest machines now under design and construction.

2. *Some remarks on two-valued logic*, by Mr. Walter Hoffman, Wayne University.

Two Boolean functions are said to be of the same type, if one can be obtained from the other by permutations of variables, or by replacing variables x_i by $x_i' = 1 - x_i$. The logical significance of this definition is discussed.

3. *Functions of measure*, by Professor B. J. Eisenstadt, Wayne University, introduced by the Secretary.

In characterizing the space $\Lambda(\phi)$, introduced in "Some new functional spaces" by G. G. Lorentz, *Ann. of Math.* (2), 1950, the following question arises. When are certain functions on a Boolean algebra, which arise naturally in the study of the space $\Lambda(\phi)$, expressible as a function of a measure on the algebra? This paper answers the question with the following theorem.

THEOREM. *If $\Phi: B \to R$ is a function from a countably complete Boolean algebra to the reals such that*

(1) *Φ is positive and increasing*

(2) *Φ is atom free*

(3) *If $e_n \in B$ and $e_n \downarrow 0$, $\Phi(e_n) \to 0$*

(4) *If $e_1 \cap e_2 = 0 = e_3 \cap e_4$ and $\Phi(e_1) \geqq \Phi(e_2)$, $\Phi(e_3) \leqq \Phi(e_4)$ then $\Phi(e_1 \cup e_3) - \Phi(e_1) \leqq \Phi(e_2 \cup e_4) - \Phi(e_2)$,*

then there exists a positive atom free, totally finite measure μ on B and a continuous, concave, increasing function F, vanishing at zero, such that $\Phi = F(\mu)$; and conversely.

4. *On using the reciprocal function for a linear interpolation*, by Professor H. E. Stelson, Michigan State University.

This paper considers conditions for determining the better inverse interpolation in cases where it is optional whether a table for $f(x)$ or a table for $1/f(x)$ is used.

5. *Power series expansions for inverse functions*, by Professor J. S. Frame, Michigan State University.

Reverting a power series that expresses w as a function of z near $z=0$ consists in finding the coefficient b_k in a corresponding series that expresses z as a function of w near $w=0$. Letting $a(z)=\sum_1^\infty a_k z^k$ and $b(w)=\sum_1^\infty b_k w^k$, we generalize the problem by setting $(w/z)^t=1+a(z)$ and solving for $(z/w)^t=1+b(z)$, assuming that $w/z=1$ at the origin. The coefficient b_k is found as the residue at $w=0$ of $(z/w)^t w^{-k-1}$, which is transformed by integration by parts to be the residue at $z=0$ of $(t/(t+k))(w/z)^{-t-k}z^{-k-1}$. Thus we obtain

$$b_k=\frac{t}{t+k}\sum_{r=0}^{\infty}\binom{-(t+k)/s}{r}a_k^{(r)} \quad \text{where} \quad [a(z)]^r=\sum_{k=1}^{\infty}a_k^{(r)}z^k.$$

This formula is applied first to find convergent series for all the roots of the general trinomial equation, and then to obtain certain known expressions for the roots of Bessel functions.

6. *A one page table of fifteen-place logarithms*, by Mr. Abraham Nemeth, University of Detroit.

The method described here depends upon the fact that any number can be decomposed into factors which converge to 1. After a number has been so decomposed, the logarithms of the individual factors can be found from a short table, constructed as follows.

The first column is headed N, and entries appear from 1 to 9 inclusive. The next column contains the logarithms of these numbers to fifteen places. The next column contains the logarithms of the numbers from 1.1 to 1.9 inclusive. The next, the logarithms of the numbers from 1.01 to 1.09 inclusive. The table proceeds in this way, the last column containing the logarithms of numbers from 1.000000001 to 1.000000009 inclusive.

The process is also reversible and it is possible to find N when $\log N$ is known.

7. *Brook Taylor, his life, work, and theorem*, by Professor P. S. Jones, University of Michigan.

Brook Taylor lived from 1685–1731 but his mathematically productive period was largely compressed into the years 1708–1715. The first year marked his determination of centers of oscillation, published in 1714, and the latter year marked the appearance of the first editions of his two books *Linear Perspective and Methodus Incrementorum*.

"Firsts" found in the latter are: an interpolation formula (a refinement of Newton's) from which Taylor's theorem is derived, the formulas for integration by parts and interchange of dependent and independent variables in differentiation, and the recognition of singular solutions in some specific differential equations together with the application of his methods to the problems of the vibrating string, the free hanging and loaded string, capillary action, the density and diffraction of the atmosphere, centers of percussion.

Poor health and the emotional disturbances due to estrangement from his austere father at the time of his first marriage and to the deaths in childbirth of both his first and second wives probably account for his cessation of mathematical activity and his turning to religious and philosophical writing in his later years. Taylor's *Contemplatio Philosophica* was published by his grandson in 1793.

It still seems fair to give Taylor credit for priority in the publication of a general formula for expanding functions into infinite series, for an independent derivation however non rigorous, and for recognizing the generality and utility of his series in spite of Johannes Bernoulli's vigorously asserted claims to priority and the recent discovery in the papers of James Gregory of several series expansions using successive derivatives.

8. *The program of the Association*, by Professor G. B. Price, University of Kansas.

The objectives and responsibilities of the Association were discussed in this invited address. Some suggestions for improved methods of attaining these objectives were made.

9. *Projectively Euclidean Hermitian manifolds*, by Professor S. I. Goldberg, Wayne University.

Let M^n (complex dimension n) be a differentiable manifold of constant holomorphic curvature k. These spaces for positive, negative and zero values of k have recently been characterized (topologically) by J. Igusa. General measures of deviation from k have been defined by S. Bochner and it was found that for deviations within certain limits the Betti numbers remain unchanged. For example, for $k>0$, the projective curvature tensor gives a measure of the deviation from complex projective space. A subgroup of the group of projective transformations for Hermitian manifolds with an asymmetric connection is defined. This leads in a natural way to the definition of a manifold $\widetilde{M}^n$ whose curvature tensor has the same formal description as that of M^n. The following results are established: (i) $\widetilde{M}^n$ has constant holomorphic curvature, if and only if, k is a constant, (ii) an $\widetilde{M}^n$ is conformal with some M^n, and (iii) if $\widetilde{M}^n$ is compact it is torsion free.

10. *Conditional covariance and light*, by Professor G. Y. Rainich, University of Michigan.

Physicists use two models describing the phenomenon of light: one is the electromagnetic field, the other is a corpuscle. Four-dimensionally the first involves a matrix M whose elements are the components of the electric and magnetic vectors. In the general case M has two invariants but in the case of light (when the two vectors are at right angles and of equal length) both invariants vanish. As a result, the square roots of the diagonal elements of the square of M behave like the components of a four-vector of zero square which is the model used in the corpuscular theory. The situation is considered from the point of view of conditional covariance.

11. *Some geometric aspects of integration*, by Professor W. F. Eberlein, Wayne University.

Connections between properties of the ordinary Riemann integral and elementary separation properties of convex sets in Hilbert space are formulated. In particular, a geometric proof of the Osgood convergence theorem is obtained. The methods generalize to more abstract spaces and integrals.

12. *The ring of $n\times n$ matrices over an AW^*-algebra*, by Mr. S. K. Berberian, Michigan State University.

Let A be an AW^*-algebra of type II_1, A_n the ring of $n\times n$ matrices over A. It is conjectured that A_n is AW^*. We show: (a) A_n is a B_p^*-algebra; (b) if A (equivalently the center of A) is of denumerable type, then A_n is AW^* of denumerable type. The main technique is the regular ring attached to the continuous geometry of projections of A.

S. D. CONTE, *Secretary*

THE MARCH MEETING OF THE OKLAHOMA SECTION

The spring meeting of the Oklahoma Section of the Mathematical Association of America was held at the University of Tulsa, Tulsa, Oklahoma, on March 31, 1956. Professor Truman Wester, Chairman of the Section, presided. There were 57 persons in attendance, including 42 members of the Association.

The following papers were presented:

1. *A note on Borel's integral method of summation*, by Professor H. N. Carter, University of Tulsa.

Borel's definition for the sum of a series based on an integral is given by the equation

$$s = \int_0^\infty e^{-a}\left(u_0 + u_1 a + u_2\frac{a^2}{2!} + \cdots + u_n\frac{a^n}{n!} + \cdots\right)da.$$

Substitute $u(a)$ for

$$u_0 + u_1 a + u_2\frac{a^2}{2!} + \cdots + u_n\frac{a^n}{n!} + \cdots$$

and $s = \int_0^\infty e^{-a}\,u(a)da$ which is Borel's integral method for determining the sum of the series $\sum_{n=0}^\infty u_n$.

The sum of any series whose terms are integral powers of $(n+1)$ can be determined by Borel's integral method by deriving a general formula for $u(a)$ corresponding to the series $\sum_{n=0}^\infty (-1)^n(n+1)^k$, $k=0, 1, 2, \cdots$. For this series

$$u(a) = 1 - 2^k a + \frac{3^k a^2}{2!} - \frac{4^k a^3}{3!} + \cdots + (-1)^n(n+1)^k\frac{a^n}{n!} + \cdots.$$

A formula can be derived to represent the constant coefficients of the terms in the right member of this equation by using the identity

$$(x-1)^r = {}_rC_0 x^r - {}_rC_1 x^{r-1} + {}_rC_2 x^{r-2} - \cdots - (-1)^v\,{}_rC_v x^{r-v} + \cdots + (-1)^r, \qquad r = 1, 2, 3, \cdots$$

Then $u(a) = \sum_{r=1}^\infty (-1)^{r-1}P_r a^{r-1}$ where

$$P_r = \sum_{v=0}^{v=r}(-1)^v\left[\frac{(r-v)^{k+1}}{(r-v)\,\lfloor v}\right], \qquad v = 1, 2, 3, \cdots.$$

This formula for $u(a)$ enables one to readily determine the sum of the series $\sum_{n=0}^\infty (-1)^n(n+1)^k$, $k=0, 1, 2, 3, \cdots$ by Borel's integral method.

2. *Another method of steepest descent in linear programming*, by Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

3. *Elements of area in coordinate-transformation*, by Professors W. A. Rutledge, M. Schwartz and Simon Green, University of Tulsa.

In vector analysis the study of general surface coordinates usually involves the element of area expressed by $(\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v)dudv$ where $\mathbf{r}$ is a position vector and u, v are orthogonal surface coordinates. Generally the development omits the use of this form in relation to finding areas or expressing area elements for a change of variables. In this paper the usefulness of this form is discussed and illustrated with examples.

4. *An algorithm for simplifying games*, by Professor Harold Shniad, University of Arkansas.

A finite two-person game in normal form is given by a finite matrix. If the value v of the game is known to satisfy an inequality of the form $v_1 \leqq v \leqq v_2$, then it is shown how the original game may be replaced by another game such that each element of the new corresponding matrix satisfies the given inequality and such that the value of the new game is the same as that of the original one. The new matrix may be used to make sharper estimates of the value of the game and to simplify the finding of dominating strategies.

5. *The Cevian chain*, by Professor N. A. Court, University of Oklahoma.

If the Cevian triangle (T_1) of the point M for a triangle (T_0) be constructed, then the Cevian triangle (T_2) of M for (T_1), and again the Cevian triangle (T_3) of M for (T_2), *etc.*, a "Cevian chain"

of triangles is obtained such that the trilinear polars of M for all the triangles of the chain coincide, say, in a line m. In each triangle of the chain a conic may be inscribed so as to be circumscribed about the following triangle of the chain. The point M and the line m are pole and polar for each conic of the chain thus obtained. This paper is scheduled to appear in full in the quarterly *Scripta Mathematica*.

6. *Vectorial analytic geometry*, by Professors Simon Green, M. Schwartz and W. A. Rutledge, University of Tulsa, presented by Professor Green.

This paper is concerned with the use of the algebra of vectors in a first course in analytic geometry. Illustrations are given which compare the traditional methods of analytic geometry with vector methods.

7. *Convergence of a simple geometric process*, by Professor J. R. Foote, University of Oklahoma.

A geometric process is presented by examples which suggest that either convergence or divergence of a certain sequence of points can be obtained. An approximating function with an iterative procedure is easily written. Elementary methods at once yield the desired results in a single theorem, a portion of which follows. Let $F_i(x)=m_ix+b_i$, $i=1, 2$, and $M=-m_1/m_2\neq 1$, and $x_0=k/(m_1+m_2)$, where

$$F_1(x^i) + F_2(x^{i+1}) = k + b_1 + b_2 \text{ defines the iteration.}$$

If $x^1=x_0$, then $x^i=x_0$ for all i and independently of M. If $x^1\neq x_0$, as $i\to\infty$ then $x^i\to\infty$, or $x^i\to x_0$, according as $|M|>1$ or $|M|<1$ respectively.

8. *Some consequences of the Brouwer plane translation theorem*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

The relation of the Brouwer plane translation theorem to some of the classical fixed point theorems is discussed. A proof that the 2-cell has the fixed point property is given using the plane ranslation theorem. Other more special fixed point theorems are discussed.

9. *Mathematical research in the petroleum industry*, by Dr. D. R. Shreve, Carter Oil Company, Tulsa, Oklahoma.

The applications of new mathematical techniques in the petroleum industry, made feasible by the development of high speed computers, was indicated. Mathematical controls were described, from use of statistically directed control of exploratory drilling following improved interpretation of geophysical data, with control of injection and production through predictions of production history of a reservoir, by calculated schedules through pipe lines and primary storage, through a refinery controlled by linear programming or other methods of operations research, with distribution to warehouses to meet predicted demands.

10. *Demonstration*, at Carter Research Laboratory, of the "IBM 650" computing machine with drum memory.

R. V. ANDREE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 16–17, 1956, at U. S. Navy Supply Corps School, Athens, Georgia. Professor R. H. Moorman, Chairman of the Section, and Professor D. F. Barrow, Vice-Chairman, presided over the general sessions; Professors S. T. Gormsen, C. G. Latimer, I. E. Perlin, C. G. Phipps, E. B. Shanks and W. L. Williams presided over subsections.

There were about 250 in attendance including 148 members of the Association.

The following officers were elected for the coming year: Chairman, Professor G. B. Huff, University of Georgia; Vice-Chairman, Professor C. G. Latimer, Emory University; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *A probability problem arising in grading*, by Professor G. B. Huff, University of Georgia.

The student's knowledge of the chronological sequence of a series of events may be tested by listing the events in a random order and asking the student to order the events. A measure of the student's answer may be obtained by adding the absolute values of the differences between the student's answer and a key. This leads to the question: What is the expected total of the absolute values of the differences between a given permutation of the numbers from 1 to n and a permutation obtained by shuffling and drawing? It is shown that the answer is $(n^2-1)/3$.

2. *Rubber stamp problem*, by Mr. W. R. Carnes, Georgia Institute of Technology.

An infinite rubber stamp prints concentric circles whose radii are all irrational. What is the least number of times that the stamp must be placed on an infinite plane so that each point on the plane will be covered by a circle? An existence-type solution is given, showing that three times is sufficient if the centers are properly chosen. Three such points are demonstrated and a proof that every point is covered by a circle is given using the distance formula and properties of irrational numbers.

3. *Cayley-Hilbert spaces*, by Professor V. D. Gokhale, Atlanta University.

Cayley-Banach and Cayley-Hilbert spaces are generalizations of the classical Banach and Hilbert spaces. They are vector spaces over Cayley's 8-unit division algebra. After defining these spaces by a suitable set of postulates, a generalization of the Neumann-Jordan Theorem on the equivalence of Cayley-Banach and Cayley-Hilbert spaces is proved.

4. *Results of a questionnaire*, by Professor Tomlinson Fort, University of South Carolina.

Professor Fort reported on the returns of a questionnaire recently sent out by the Committee on Personnel and Education of the Association.

5. *A history of the Southeastern Section*, by Professor H. A. Robinson, Agnes Scott College.

Immediately after the formation of the Association, December, 1915, the late Dean R. P. Stephens attempted to organize the Section; however, organization did not occur until 1922. For the 31 annual meetings held, 34 guest mathematicians have been sponsored. In first eleven years there was an average of eight papers presented with 49 present including 18 Association members; since 1949, 39 papers, 193 present including 111 members.

6. *The use of analogy*, by Professor C. G. Phipps, University of Florida.

Reasoning by analogy permits us to transfer our training from one situation to an analogous one. Thus analogy yields alternate interpretations of deductive systems since two perfectly analogous situations have the same mathematical formulas. Algebra and geometry being analogous combine into analytic geometry.

7. *Mathematics and philosophy*, by Professor R. H. Moorman, Tennessee Polytechnic Institute.

The author traced the historical relationship of mathematics and philosophy in the Greek period, the middle ages and modern times. The type of analysis which Plato introduced into mathematics was introduced into philosophy by Descartes. The type of analysis which Descartes introduced into mathematics has been used successfully in philosophy only in the twentieth century in the field of symbolic logic. The concept of "universal mathematics" goes back as early as Pythagoras. Utilitarian sciences develop by the mathematization of philosophy. Certain modern pseudo-sciences do not have the validity they claim because they do not have functional relationships.

8. *Non-associative number theory*, by Professor Trevor Evans, Emory University.

On replacing the unary operation of successor in Peano's postulates for the natural numbers by a binary operation, a characterization is obtained of a non-associative arithmetic studied by Etherington. The number theory of this system is investigated and Fermat's Last Theorem proved in it. In a manner analogous to the embedding of the natural numbers in the ring of integers, this non-associative arithmetic is embedded in the left neoring recently studied by Bruck. The number theory of this left neoring is also studied in some detail.

9. *Rational univalent functions*, by Professor W. C. Royster, Alabama Polytechnic Institute. (By title.)

Let F be the class of functions $z+\sum_{n=2}^{\infty} a_n z^n$ which are univalent in the unit circle $E(|z|<1)$. Linis (this MONTHLY, vol. 62, 1955) gave a short proof of a theorem of Friedman (*Duke Math. Journal*, vol. 13, 1946) which states that if the a_n are rational integers then $f(z)$ is one of nine rational functions. The purpose of this note is to extend the method of Linis to quadratic fields with negative discriminant and to give criteria for selecting combinations of possible coefficients yielding rational functions which are regular and univalent in E.

10. *A note on the solution of algebraic equations*, by Professor Stephen Kulik, University of South Carolina.

A method is derived for computing by approximation any root of an algebraic equation, all the roots of which are real. It gives the lower and upper bounds of the root converging to it as a limit.

11. *Consistency of equations*, by Professor J. M. Thomas, Duke University. (By title.)

This paper points out the usefulness in teaching elementary differential equations of de la Vallée Poussin's condition (*Ann. Soc. Sci. Bruxelles*, vol. 64, 1950, pp. 74–75) for a differential whose coefficients are only assumed continuous, to be exact, and the possibility of using the idea to get a consistency (passivity) condition for a system of total differential equations under the same assumption.

12. *A note on lattice-ordered rings*, by Professor R. G. Blake, University of Florida.

Nearly ten years ago Garrett Birkhoff wrote "The theory of lattice-ordered rings . . . is not yet even in its infancy." The statement seems to be still true today. This paper suggests a set of axioms for a type of lattice-ordered ring which seems to promise some interesting results.

13. *The free lattice generated by a set of chains*, by Dr. H. L. Rolf, Vanderbilt University.

The free lattices $FL(2+2)$ and $FL(4+1)$ are shown to be infinite and their structures determined. It is shown that $FL(3+2)$ and $FL(5+1)$ each contains a sublattice isomorphic to $FL(1+1+1)$. Furthermore, $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n+m)$.

14. *Wide open sets*, by Professor D. F. Barrow, University of Georgia.

Professor Barrow defines a wide open set of numbers as a set such that the sum of two numbers of the set is never a number of the set; and a complete wide open subset of a set S, as a wide open subset of S to which no other number of S could be adjoined without destroying its wide open property. (For example, the odd integers are a complete wide open subset of the integers.) A few simple theorems were presented.

15. *Matrices with elements in a Boolean ring*, by Professor A. T. Butson, University of Florida.

Let $\mathfrak{B}$ be a Boolean ring of at least two elements containing a unit 1. Form the set $\mathfrak{M}$ of matrices $A, B, \cdots$ of order n having entries $a_{ij}, b_{ij}, \cdots, i, j = 1, 2, \cdots, n$, which are members of $\mathfrak{B}$. For any matrix A of $\mathfrak{M}$ there exists a unimodular matrix U of $\mathfrak{M}$ such that $UA = H$ has the following properties: $h_{pq} = 0$ for $q > p$, $h_{pq}h_{qq} = 0$, and $h_{pq}h_{pp} = h_{pq}$ (note that if a diagonal element is 0, then the entire row consists of 0's). This form H is unique.

16. *Remarks on Boolean functions II*, by Dr. D. O. Ellis, Research Engineer, National Cash Register Electronic Division, Hawthorne, California.

It is shown that certain of the algebraic functions of two variables in a Boolean algebra give rise to Boolean algebras centered on given elements of the original algebra. The relations among the operations of these algebras are determined explicitly. A criterion is given for the possession by any Boolean function (of a finite number of variables) of a linear factor in one of its variables. The methods employed are elementary.

17. *Some results on a particular non-associative algebra*, by Professor Tom Baynham, Jr., University of Tennessee.

In considering a non-associative algebra A^* defined by G. C. Holt in his paper entitled "A non-associative algebra generated by square matrices," (*Tennessee Academy of Science Journal*, 27: 281–290, 1952), results are obtained concerning nilpotent elements, idempotent elements, ideals and the center of A^*.

18. *An orthotropic beam bent by its own weight*, by Professor C. B. Smith, University of Florida.

An infinitely long beam is considered to be in a state of plane stress under the action of its own weight and equal supporting reactions applied at a series of points situated at regular intervals along its lower edge. By considering various portions of the beam, it is possible to draw some conclusions concerning the bending of finite beams loaded by gravity when the ends are either simply supported or clamped.

19. *Application of shallow shell theory to the buckling of spherical shells*, by Professor W. A. Nash, University of Florida.

The linear differential equations governing the axi-symmetrical buckling of shallow spherical shells subject to uniform external pressure are developed for the case of small deformations. An approximate solution to these equations, based upon Galerkin's method, is presented for the case

of a simply supported spherical cap and a displacement pattern suitable for use with clamped edge spherical caps is indicated. The results obtained indicate the potentialities of shallow shell theory for use in approximating classical small deformation buckling loads of shells which are not shallow.

20. *A comparison of the Deming and Garwood fitting methods*, by Mrs. Nancy M. Dismuke, Oak Ridge National Laboratory.

The Deming and Garwood procedures for fitting data to a function non-linearly dependent on a set of parameters have been applied to several problems. Based on this experience a comparison is made of the convergence properties, the parameter variances obtained, and the labor involved in applying the two methods.

21. *Stress distribution in an orthotropic disk subjected to its own weight and supported by a concentrated force on its boundary*, by Professor J. B. Wilson, University of Florida.

An orthotropic disk subjected to its own weight and supported by a concentrated force applied at its boundary is considered as the limit of the case in which the supporting force is distributed over an arbitrarily small area on the boundary. Series solutions for the stresses and displacements are obtained, and the stress function for the orthotropic disk is shown to contain as a limiting case the stress function for an isotropic disk. The stress components are calculated at several points along the horizontal diameter of a plane-sawn disk of Sitka spruce, with the grain taken to be horizontal.

22. *Curvatures associated with a vector field*, by Professor J. D. Novak, University of South Carolina.

In this paper W. C. Graustein's associate curvature and T. K. Pan's normal curvature for a vector field in a surface are generalized for any vector field associated with the surface. These generalized curvatures are used in the study of curves on the surface.

23. *On picture writing*, by Dr. George Pólya, Visiting Lecturer, Mathematical Association of America. (By invitation.)

In several combinatorial problems we can arrive more intuitively at the expression, or defining equation, of the *generating function* that solves the problem if, instead of the letters of various alphabets, we use appropriate pictures as symbols for variables or indeterminates. This idea, which the author has used in research for more than twenty years but has not published before, is illustrated by three examples in the present lecture which is scheduled to appear in the MONTHLY.

24. *The elementary derivatives from a geometrical definition of e*, by Professor J. T. Moore, University of Florida. (By title.)

In the class of curves $y=a^x$ define e to be that value of a such that $y=a^x$ crosses the y-axis with a slope of 1. By definition, then, $D_x e^x=1$, *for* $x=0$. An application of the Δ-process then leads to $\lim_{\Delta x\to 0}(e^{\Delta x}-1)/\Delta x=1$ and $D_x e^x=e^x$ *for all* x*'s*. By writing $y=\ln x$ and $y=x^n$ in the respective forms $x=e^y$ and $\ln y=n \ln x$, the log and power function formulas are obtained immediately. The following advantages are claimed: (1) binomial theorem is not needed; (2) no complex identities or inductive methods are used; (3) definition of e is more understandable than the usual algebraic definition as a limit.

25. *Understanding mathematical induction*, by Professor R. W. Ball, Alabama Polytechnic Institute.

Mathematical induction can be taught successfully if it is related to reasoning processes the student has already used. The instructor should require, before the formal proof, a thorough dis-

cussion of the statement of the theorem, a specific example of the extension process, and a conclusion for some finite case. Only after the student is convinced that such reasoning does not depend on n should he attempt the general proof.

26. *A note on degenerate conics*, by Professor J. W. Lasley, Jr., University of North Carolina.

It is well known that a real conic degenerates into a pair of lines if the rank of its matrix is 2; into a repeated line if its rank is 1. If the lines are distinct, they are real if the 2×2 determinant in the upper left hand corner is minus; conjugate imaginaries, if this sign is plus. In the theory of real quadratic forms these criteria may be made in terms of the rank of the adjoint matrix and the sign of its trace. This paper reviews the theory of degenerate conics from this higher viewpoint.

27. *Partial fractions*, by Professor L. A. Dye, The Citadel.

The resolution of a proper rational fraction into a set of simpler fractions is demonstrated by evaluating the necessary parameters for the general case. Some examples of special methods for particular cases are given.

28. *Georgia Institute of Technology Computer Program—a progress report*, by Professor B. M. Drucker, Georgia Institute of Technology.

The School of Mathematics of Georgia Institute of Technology initiated a curriculum in digital computers and numerical analysis in the fall of 1954. The Rich Electronic Computer Center of the Experiment Station was activated in the fall of 1955. This report comprises comments as to future plans in light of the experience obtained to date and suggestions to other departments interested in setting up such a program.

29. *Locally one-to-one mappings on S^1*, by Professor M. K. Fort, Jr., University of Georgia.

Let f be a mapping on the unit circle S^1 into a one dimensional metric space Y. If f is locally one-to-one on S^1 (*i.e.*, each point of S^1 has a neighborhood on which f is one-to-one), then f is an essential mapping.

30. *A note on the matrix equation $AX=Xf(A)$*, by Professor R. H. Ackerson, University of Florida.

The matrix equation $AX=XB$ has been considered by W. V. Parker, with A in rational canonical form. Using Parker's method, a necessary and sufficient condition was determined for the equation $AX=Xf(A)$ to have all of its solutions expressible as polynomials in A, where A is non-derogatory, and $f(A)$ is a polynomial in A.

31. *On the simple group of order 5616*, by Professor F. A. Lewis, University of Alabama.

This paper gives some results of a study of $LF(3,3)$ in terms of generators corresponding to elementary operations used in reducing a determinant.

32. *On the rational equivalence of ternary quadratic forms*, by Mr. Amin Muwafi, University of Florida.

The first part is a method of obtaining a linear rational transformation, whose determinant has a given value $C\neq0$, that takes a ternary quadratic form with rational coefficients into one with rational coefficients and without cross-products. Formulas are given by which the three columns of the matrix of the transformation are determined. In the second part, sufficient conditions are found in order that a given ternary quadratic form, of determinant $d\neq0$, may be taken into an-

other given ternary quadratic form of the same determinant d by a linear rational transformation of determinant one.

33. *A note on the distribution of logical quantifiers*, by Professor R. S. Fouch, Florida State University.

Quine has proved a number of theorems, grouped into six chains, concerning the distribution of the universal and existential quantifiers over three truth-functions. However, there are 13 other truth-functions, of which 7 are non-trivial. Similar distributive theorems for these 7 functions are established by a simple decision method. A structure is determined for the theorems for each function and it is shown that there are only 3 different structures for the 10 functions. The interrelationships of these structures are also examined.

34. *Note on perturbations of systems having a periodic solution*, by Professor J. A. Nohel, Georgia Institute of Technology.

Let x, f be real vectors with n components and suppose t and μ are real. Consider the system (1) $x' = A(t)x + \mu f(t, x, \mu)$, $(' = d/dt)$, where $A(t)$ is a real periodic $n \times n$ matrix with period T and f has period 2π in t. Suppose that for $\mu = 0$ the linear system (2) $x' = A(t)x$ has at least one solution with period 2π. Sufficient conditions are given for the existence of periodic solutions of (1) for $|\mu|$ sufficiently small. This generalizes results of Coddington and Levinson (*Contributions to the Theory of Nonlinear Oscillation, vol. II*, Annals of Math. Studies no. 29, Princeton University Press) where $A(t)$ is a constant matrix.

35. *On the behavior of solutions of ordinary finite difference equations*, by Mr. A. C. Downing, Jr., Oak Ridge National Laboratory.

It is well known that when finite difference equations are substituted for differential equations, it does not follow *a priori* that the behavior of the solution of the difference equations will approximate the solution of the differential equations. This is demonstrated for the very simple ordinary differential equation $y'' - \lambda^2 y = 0$. When λ^2 is real, the character of the linearly independent solution depends upon whether $\lambda^2 > 0$, $\lambda^2 = 0$ or $\lambda^2 < 0$. The solutions of the corresponding finite difference equation (as usually formulated) fall into four categories. The anomalous character of the fourth category is readily recognized.

36. *Solution of a non-linear partial differential equation of the third order*, by Professor R. W. Cowan, University of Florida.

The type of third order equation to be solved is determined by the form selected for the intermediate integral. Then equations analogous to Monge's equations are derived and used to determine intermediate integrals. From the latter a general solution may sometimes be found. If this is not possible other types of solutions can be determined by specializing the arbitrary functions occurring in the intermediate integrals.

37. *Basic sets of polyharmonic polynomials*, by Professors E. P. Miles, Jr. and Ernest Williams, Alabama Polytechnic Institute.

The authors show that the four homogeneous polynomials of degree n, $n \geqq 4$,

$$B_\alpha^n(x, y) = \sum_{j=0}^{[(n-\alpha)/2]} (-1)^j (j+1)^{[\alpha/2]} \frac{n(n-\alpha)(n-\alpha-1)\cdots(n-\alpha-2j+1)}{(2j+\alpha)!} x^{n-\alpha-2j} y^{\alpha+2j} \quad (\alpha = 0,1,2,3)$$

constitute a basic set of solutions for the biharmonic equation, $\nabla^4 xy(u) = 0$; for $\alpha = 0, 1$, the polynomials are the real and imaginary parts of $(x+iy)^n$. These results are generalized to obtain a basic set of solutions for $\nabla^4_{z_1, z_2} \cdots {}_{z_k}(u) = 0$; this set contains the $C_{n+k-3,\ k-3}(2n/(k-2)+1)$ harmonic polynomials of the authors' basic set (*Proc. A.M.S.*, vol. 6, 1955) for $\nabla^2_{z_1, z_2, z_3, \cdots, z_k}(u) = 0$. A further generalization gives a basic set for $\nabla^{2k}_{z_1, z_2, z_3, \cdots, z_k}(u) = 0$.

38. *Periodic solutions of a non-linear wave equation*, by Professor F. A. Ficken, University of Tennessee.

Define $Lu=u_{tt}+2ku_t+au-u_{xx}$. We know that the problem $Lu=b(x,t)-\epsilon u^3$, $u(x,0)=f(x)$, $u_t(x,0)=g(x)(-\infty<x<\infty)$ has a unique solution $u(f,g;x,t)$ if $|\epsilon|$ is properly restricted. Here we give a new and simpler proof of a known further fact: if $b(x,t+p)=b(x,t)$, $k>0$, and $a>0$, then for certain f_1 and g_1, u is also p-periodic in t ($u(f_1,g_1;x,t+p)=u(f_1g_1;x,t)$). This solution is now seen to be, as in the linear one-dimensional case, a steady motion that is approached, as transients disappear, by any $u(f,g;x,t)$ meeting certain natural conditions. Special cases: $0\leq x$ and $u(0,t)=0$; $0\leq x\leq\pi$ and $u(0,t)=0=u(\pi,t)$. All this work has been done in collaboration with B. A. Fleishman.

39. *From matrices to tiles*, by Dr. T. W. Hildebrandt, Oak Ridge National Laboratory.

Property (A) for matrices was defined by D. Young (*Trans. A.M.S.* vol. 76, 1954, pp. 92–111). For the matrix of the finite difference representation for a linear elliptic partial differential equation, this property has a simple geometrical interpretation, related to the problem of "tiling" [filling the plane (space) with congruent polygons (polyhedra)].

40. *The class of integral sets*, by Professor E. B. Shanks, Vanderbilt University.

The class of integral sets was defined in terms of the non-negative integers, using the binary system. It was shown that this system satisfies the axioms of set theory used by Gödel, provided the axiom of infinity postulates an infinite class instead of an infinite set. A relation that partially orders the system was discussed also.

41. *Fixed points for limit functions*, by Professor W. L. Strother, University of Miami.

That the pointwise limit of continuous functions is not necessarily continuous is well known. A new sort of limit is defined which agrees with the pointwise limit at all points of continuity of the latter and is weakly continuous. For certain spaces which have fixed point property for continuous functions it is shown that the limit of continuous functions has a fixed point.

42. *The effect of symmetrization on smoothness of functions*, by Professor W. S. Snyder, University of Tennessee.

The function $f(x,y)=f(r)=r^4(1+\cos\pi/r)$ has continuous first partial derivatives on $0\leq x^2+y^2=r^2\leq 1$, but its symmetrization with respect to the origin is not piecewise smooth. (For a definition of symmetrization, see Pólya, G. and Szegö, G., *Isoperimetric inequalities in mathematical physics*, Annals of Math. Studies, no. 27, pp. 151–152).

43. *A topological equivalence theorem*, by Professor R. A. Lytle, University of South Carolina.

Garrett Birkhoff shows in his paper, *The topology of transformation sets*, Annals of Mathematics, vol. 35, that the point open topology is equivalent to the uniform topology for the space of homeomorphisms of the unit interval onto itself. This paper generalizes this result as follows: Suppose A is a graph which is a continuum and H is the set of all homeomorphisms of A onto A. Then H with the overlap topology is equivalent to H with the uniform topology.

44. *A note on the Cauchy criterion*, by Mr. G. E. Duncan, Georgia Institute of Technology.

This paper demonstrates the sufficiency of the Cauchy criterion by a direct application of the nested set theorem.

45. *The new freshman mathematics*, by Professor J. D. Mancill, University of Alabama.

What is bringing about change in our first year college mathematics? Mathematics has fanned out in ever-increasing new directions with the effect that neatly compartmentalized subjects have become mixed beyond recognition. Importance of development is having its effect on curriculum in secondary schools and freshman college levels. We are confronted with the problem of putting elementary mathematics back in the center of the ring of "general education." It must be realized that to revitalize the position of elementary mathematics in the light of new developments involves change, or at least the willingness to change.

46. *Universal mathematics*, by Professor C. L. Seebeck, Jr., University of Alabama.

Universal mathematics course designed by the Committee on the Undergraduate Program of the Association was outlined and discussed. Report was favorable for the experimental course given at University of Alabama.

47. *General or unified mathematics*, by Professor F. W. Kokomoor, University of Florida.

Students having varied preparational backgrounds and taking mathematics only to obtain what "every college graduate ought to have" find integrated courses better than the traditional, which are mainly preparatory for something they will never reach. However, some of these students do become intensely interested and turn out to be excellent mathematicians. Integrated courses can also better serve majors in other fields by giving prerequisite material earlier and omitting unnecessary content. Majors in mathematics itself lose nothing by taking the integrated courses, but do just as well with the traditional, if they have no prerequisite deadlines to meet along the way.

48. *The postulational approach to freshman mathematics*, by Professor Herman Meyer, University of Miami.

The postulational approach to freshman mathematics taught at the University of Miami for the past seven years is described. Some examples of the methods, proofs, and procedures used are set out, along with a subjective analysis of the strength and weakness of the program.

49. *Some remarks on the location of the roots of an algebraic equation*, by Professor A. T. Brauer, University of North Carolina.

It is well known that every theorem on characteristic roots of a matrix contains a theorem on algebraic equations as a special case. It is shown that the following known theorems can be obtained easily by using the columns and the rows of the same matrix. (See O. Perron, *Algebra*, 2nd ed., Berlin, 1933, pp. 31–33.) Let r be the absolute greatest root of the equation $x^n+a_1x^{n-1}+\cdots+a^n=0$, and $m, m_1, m_2, \cdots, m_{n-1}$ arbitrary positive numbers. Then,

(I) $|r| \leqq \max\left(|a_1| + m_1^{-1}, |a_2| m_1 + m_2^{-1}, \cdots, |a_{n-1}| m_1 m_2 \cdots m_{n-2} + m_{n-1}^{-1}, |a_n| m_1 m_2 \cdots m_{n-1}\right)$;

(II) $|r| \leqq \max\left(m^{-1}, |a_1| + |a_2| m \cdots + |a_n| m^{n-1}\right)$.

Moreover, improvements of these results are obtained.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, on March 17, 1956. Professor S. E. Urner, Chairman of the Section, presided. The attendance was 107 including 69 members of the Association.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor H. F. Bohnenblust, California Institute of Technology; Vice-Chairman, Professor P. B. Johnson, Occidental College; Secretary-Treasurer, Professor P. H. Daus, University of California at Los Angeles; Program Committee: Professor A. R. Harvey (Chairman), San Diego State College; Professor V. C. Harris, San Diego State College; Professor R. A. Dean, California Institute of Technology; Mr. R. E. Horton, Los Angeles City College.

The following program was presented:

1. *Differentiation in vector spaces*, by Dr. Walter Noll, University of Southern California, introduced by Professor D. V. Steed.

Originally, differentials were thought of as "infinitely small" increments. Since no precise meaning can be given to the "infinitely small," they are now regarded as linear relations between arbitrary finite increments $dx=\xi$ and corresponding increments $dy=\eta=f'(x)\xi$. Such a relation can be written also in the form

$$\eta = \nabla f(x)[\xi] = \frac{d}{d\epsilon} f(x+\epsilon\xi)\Big|_{\epsilon=0}.$$

This definition can be extended in a straightforward manner to scalar or vector fields simply by replacing both x and ξ by vectors. The "differential" thus obtained is a linear form or a linear transformation. By referring the underlying space to arbitrary curvilinear coordinates, the differential of a vector field can be shown to be identical to what is usually called the "covariant derivative".

2. *Rearrangements of infinite series*, by Professor V. L. Klee, University of Washington.

The speaker reviewed some of the principal results concerning rearrangement of infinite series of vectors in Euclidean n-space, especially the basic Levy-Steinitz theorem and a closely related result due to Jarnik, Behrend, and Wald. A proof of the latter was presented, and an outline of a proof of the former. Some results of Hadwiger were also discussed.

3. *College mathematics for non-science students*, a panel discussion by Professor May M. Beenken, Immaculate Heart College; Professor P. B. Johnson, Occidental College; Mr. R. B. Herrera, Los Angeles City College; Dr. C. J. A. Halberg Jr., University of California, Riverside.

This was a panel discussion by members of a special committee of a subcommittee on mathematics appointed by the California Committee for the Study of Education. The chairman, Professor May M. Beenken, explained the history and purpose of the committee. Professor P. B. Johnson discussed current practices in the field of mathematics for general education as gleaned from a questionnaire study made by the committee. Mr. R. B. Herrera spoke on textbooks and literature

in this field of mathematics and distributed an annotated bibliography compiled by the committee. Dr. C. J. A. Halberg, Jr., gave the committee's recommendations.

4. *The role of the mathematician in industry*, by Mr. Ross Miller, Northrop Aircraft Corporation, introduced by Professor Clifford Bell.

The profit motive is the underlying force in industry. A corporation makes money by pleasing its customers, by satisfying their wants through better products made more rapidly, more reliably, more cheaply, etc. Frequently it creates in its customers new desires which industry is in a position to satisfy. And we note in this case the opportunity for basic research.

The mathematician's goals are those of job satisfaction and interest, a feeling of challenge and responsibility. Pleasant working conditions with adequate financial rewards are also important considerations.

The goals of industry and the goals of the mathematician can be compatible and can find fulfillment in a merger of interests. The mathematician's contribution is basic and becomes ever increasing in its importance. Industry, on the other hand, stands prepared to offer the mathematician the personal rewards which are so important to the individual.

5. *New developments in numerical analysis*, by Dr. G. E. Forsythe, Numerical Analysis Research, University of California at Los Angeles.

Automatic digital computing machines, virtually unknown ten years ago, continue breathtaking developments in speed, capacity, quantity manufactured, and demands for numerical analysis. Some 20 university departments use such machines regularly as a research tool. Academic mathematics seems to contact machines in four ways. (1) Huge mathematical experiments are carried out on such problems as Fermat's last theorem. (2) Mathematics is a principal tool in analyzing computing methods and in devising new ones. (3) Computers and their use raise many new logical and mathematical problems undreamed of before computers ran. (4) Such machines are a boon to teachers in advertising and glamorizing mathematics, and they can be understood by all who know elementary arithmetic.

6. *The theory of games and the game of poker*, by Dr. Rodrigo Restrepo, California Institute of Technology, introduced by Professor H. F. Bohnenblust.

In the last few years considerable effort has been devoted to the construction of economic, tactical or gambling models that can be analyzed by means of the theory of games. Using a simplified version of the game of poker, the author constructs a class of games in which each player must decide how much to bet. The problem is solved completely, and the results show that the familiar technique of "bluffing" is a necessary element of any optimal strategy.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The thirtieth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Geneva College, Beaver Falls, Pennsylvania, on April 28, 1956. Dean L. T. Moston, Chairman of the Section, presided during both the morning and afternoon sessions.

There were 59 persons present, including 41 members of the Association.

The Secretary reported that in accordance with the wishes of the members involved and with the approval of the Executive Committee of the Section the Board of Governors of the Association had voted to transfer Cabell County, West Virginia, from the Allegheny Mountain Section to the Ohio Section.

The following papers were presented:

1. *Semantics in information theory*, by Professor Evan Johnson, Jr., Pennsylvania State University.

It was pointed out that Warren Weaver defined the semantic problem of communication theory as that concerned with the precision with which transmitted signals convey the desired meaning. Meaning is assigned by the individual to either a single communication signal or to a set of signals, each of which is a continuous function of time. A first problem of semantics, therefore, is the definition of the set of all functions of time which are distinguishable by the individual, and to which meaning can potentially be assigned.

2. *A generalization of the game of Nim*, by Professor Donald C. Benson, Carnegie Institute of Technology.

A class of two-person games was considered which includes "Nim" and "Kayles" as special cases. (Definitions of "Nim" and "Kayles" can be found in Ball's *Mathematical Recreations*.)A criterion was given to determine the winning and losing positions.

3. *Games of evasion*, by Professor L. E. Dubins, Carnegie Institute of Technology.

The speaker discussed an interesting class of games introduced by R. P. Isaacs. He sketched some solutions and mentioned some unsolved problems.

4. *Geology as a source of problem material in elementary mathematics*, by Professor E. A. Sturley, Allegheny College.

Many of the students in undergraduate courses do not have mathematics or physics as primary interests. These courses should include more problem material from other areas to help these students relate the mathematics course to their own fields. Some geological applications of mathematical ideas were presented as illustrations.

5. *Coordination with the secondary school and possible secondary curricular changes*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper suggests curtailing the geometry courses, specially solid, and filling in with more algebra and analytic trigonometry. Personal contacts, luncheons, meetings of college and high school mathematics teachers are also advised.

6. *Computers in business and science*, by Mr. L. P. Rosenberry, I. B. M. Corporation, Detroit, Michigan.

The history of digital computer usage in commercial and engineering applications was reviewed. Several examples of present and past problems successfully solved were presented. Future plans for use of computers by several companies were indicated—linear programming, inventory control, marketing, plant design, process control, scientific calculations. Some general prognostications were presented.

I. D. PETERS, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-third regular meeting of the Iowa Section of the Mathematical Association of America was held at Grinnell College, Grinnell, Iowa, on April 20–21, 1956.

The attendance was 55, including 29 members of the Association.

The following officers were elected: Chairman, Professor F. W. Lott, Iowa State Teachers College; Vice -Chairman, Professor A. H. Blue, Cornell College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following motion carried unanimously:

WHEREAS, the cooperation of the high-school and college teachers of mathematics in Iowa is deemed to be of vital concern to the members of the Iowa Section of the Mathematical Association of America.

NOW THEREFORE, BE IT RESOLVED that the Iowa Section of the Mathematical Association of America hold, each year, a meeting in addition to the annual Spring meeting to be devoted to the pedagogic interests of collegiate mathematics and held in conjunction with a meeting attended by large numbers of Iowa high-school teachers of mathematics.

The following papers were presented:

1. *Pascal's arithmetical triangle*, by Professor R. B. McClenon, Grinnell College.

Professor McClenon presented Pascal's construction of the arithmetical triangle, and several theorems which Pascal proved by mathematical induction. It was pointed out that they, as well as the well-known applications which Pascal made, could be proved easily by using modern notation.

2. *An application of the functional equation* $f(x+y)=f(x)f(y)$ *in elementary differential equations*, by Professor L. E. Pursell, Grinnell College, and Professor R. F. Reeves, Ohio State University.

It is well known that if $y(x)$ satisfies the differential equations $dy/dx=ky$, then $r(x)=y(x)/y(0)=\exp(kx)$ satisfies the functional equation $r(x_1+x_2)=r(x_1)r(x_2)$; and, conversely if $r(x)$ is differentiable at $x=0$ and satisfies this functional equation, then for any choice of $y(0)$ the function $y(x)=y(0)r(x)$ satisfies the differential equation $dy/dx=[r'(0)]y$. In some classroom applications in which this differential equation is involved, it is more convincing to the students to demonstrate first that the functional equation is satisfied. The window problem as given in section 3.3 of *Differential Equations* by Agnew is solved as an example.

3. *Common elements*, by Professor F. A. Brandner, Iowa State College.

A rigorous and complete treatment of the subject, "Common Elements", is not to be found in any text-book of statistics, and almost no research has been done in that direction. Several definitions and their equivalence are discussed. The elementary idea is extended. Properties of common elements equations and their possible uses are developed.

4. *Charts for zeros of cross product Bessel functions*, by Professor Don Kirkham, Iowa State College.

In an earlier report (this MONTHLY, vol. 59, 1952, p. 501) some methods for obtaining the zeros of $J_n(x_p)Y_m(kx_p)-J_m(kx_p)Y_n(x_p)$ and of some similar cross products having the inner pair of terms, or all four terms differentiated were presented. Now, utilizing these methods, a series of large graphs (charts) of x *vs.* kx, from which the zeros of the cross products, with and without the primes, for $n=m=0, 1, 2, 3$, $p=1, 2, \cdots 9$, $0\leq k\leq\infty$, $k=1$ excluded, can be read to about two

decimal accuracy, has been prepared. The zeros of $J_n(x)$ and of $J_n'(x)$ lie on the graphs and are seen to be closely enough disposed to enable one to draw, to about two decimal accuracy, except for x or kx near zero, graphs of x *vs.* kx from a set of zeros of $J_n(x)$ and/or of $J_n'(x)$ alone. Zeros of $Y_n(x)$ and of $Y_n'(x)$ also lie on the graphs and are identified. The charts are of further particular interest in that they provide approximate values of x_p for large k, values which are not tabulated in the literature; show clearly that a zero of the cross products exists at $k=\infty$ and $k=0$; and show that a zero at $k=\infty$ or $k=0$ is a zero of either $J_n(x)$ or of $J_n'(x)$.

5. *History of the first forty years of the Iowa Section of the Mathematical Association of America*, by Professor Fred Robertson, Iowa State College.

The author discussed the formation of the section on April 28, 1916, in Des Moines. The names of the charter members were given. The minutes of the first meetings were reviewed. The fact that the section has met jointly with the Iowa Academy of Science since its founding and met with the State Teachers Association for the first five years of its existence was emphasized in the discussion of time, and place of its meetings. Changes over the years were shown and hopes for future progress expressed.

6. *Remarks on commuting automorphisms of rings*, by Professor M. F. Smiley, State University of Iowa.

Nathan Divinsky (*Trans. Royal Soc. Canada*, Sec. III, (3), vol. 49, 1955, pp. 19–22) discusses automorphisms T of an associative ring A which satisfy $x \cdot x^T = x^T \cdot x$ for every x in A. The present note frees this discussion from the chain condition employed by Divinsky by using results of the Jacobson structure theory of rings.

7. *A characterization of a certain type of distribution*, by Professor R. V. Hogg, State University of Iowa.

If $x_1, x_2, \cdots, x_n$ be n ordered, initially independent observations of a random variable having a frequency of the continuous type on the interval $(0, b)$, then a necessary and sufficient condition that x_j and x_i/x_j, $i<j$, be stochastically independent is that the frequency be of the form cx^{c-1}/b^c, $0<x<b$, zero elsewhere, where $c>0$.

8. *A certain limiting distribution*, by Professor A. T. Craig, State University of Iowa.

Let $F(x)$ denote the c.d.f. of a random variable x and $G_n(\bar{x})$ that of the arithmetic mean $\bar{x}$ of a random sample of n values of x. It is well known that if $E(x)=\mu$ is finite, the sequence of distributions $G_n(\bar{x})$ converges to the distribution

$$G(\bar{x}) = 0, \quad \bar{x} < \mu,$$
$$= 1, \quad \bar{x} \geq \mu,$$

at every continuity point of $G(\bar{x})$. If $E(x)$ does not exist, little is known concerning the limit of the sequence $G_n(\bar{x})$. For the class of distributions having

$$F(x) = \int_{-\infty}^{1} \frac{cy^{2[p-1]}}{(1+y^2)^p}\, dy,$$

p a positive integer, it is proved that the sequence of distributions $G_n(\bar{x})$ converges to the distribution

$$G(\bar{x}) = \frac{1}{\pi} \int_{-\infty}^{\bar{x}} \frac{K}{(K^2+t^2)}\, dt.$$

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 28, 1956 at the University of Kentucky, Lexington, Kentucky. Professor V. F. Cowling, Chairman of the Section, presided at both the morning and afternoon sessions.

There were 52 persons in attendance, including 41 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor G. G. Roberts, Berea College; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor J. C. Eaves, University of Kentucky.

By invitation of the Committee, Professor Fred Ficken of the University of Tennessee delivered an hour address at the afternoon session on "Isolated Singularities of Continuous Vector Fields in the Plane." An abstract of this address follows.

This paper was an expository account of the elementary classical theory. The behavior near the origin of streamlines of fields $(ax+by,\ cx+dy)$, $(ad-bc \neq 0)$ was treated in detail by reduction to familiar canonical forms (nodes, pass, focus, and center). After brief remarks on perturbation of these fields, the speaker introduced for any continuous field the class Γ of simple closed sectionally smooth curves not passing through a singularity of the field, and discussed the isotopy classes of Γ. Finally, the standard results on the Poincaré index of $\gamma \in \Gamma$ with respect to the field were presented.

The following papers were presented:

1. *A criterion for the irreducibility of a polynomial*, by Professor A. W. Goodman, University of Kentucky.

This was an expository talk based on a result due to Perron, *Jour. fur die Reine und Ang. Math.* vol. 132, 1907, pp. 288–307.

2. *The comparability of cardinal numbers*, by Mr. K. E. Stoll, University of Kentucky.

The usual proof of the important theorem on the comparability of cardinal numbers depends on the elementary theory of ordinal numbers and the well ordering theorem. In this paper a proof of the comparability theorem was given without introducing ordinal numbers, and without direct use of the well ordering theorem. The proof did require the axiom of choice.

3. *Gibbs phenomenon for Taylor means*, by Mr. Joseph Cornelison, University of Kentucky.

The preservation of Gibbs Phenomenon under various linear transformations was discussed, including recent work by O. Szasz [*Acta Scientiarum Mathematicarum*, vol. 12, 1950, pp. 107–111]. New results on "Gibbs Phenomenon for Taylor Means" were stated and the method of proof outlined.

4. *The transcendence of e*, by Mr. Richard Sprague, University of Kentucky.

The speaker presented a proof of the transcendence of e as found in Edmund Landau, *Vorles-*

ungen uber Zahlentheorie, vol. 3, 1927, Leipzig, p. 90+, pointing out the relatively elementary nature of the necessary ideas.

5. *Some remarks on polynomials*, by Mr. Garnett Stephens, University of Kentucky, introduced by Professor V. F. Cowling.

Descartes' rule provides an upper bound on the number, N, of real zeros of a polynomial, $P(x)$, with real coefficients. Using a generalization of a lemma of Fejer [*C.R. Acad. Sci. Paris*, vol. 158, 1914, pp. 1328–1331], a criterion for obtaining a lower bound on N is obtained based on the Tchebychef polynomial representation of $P(x)$. The method is similar to that used by Turan [*Bull. Amer. Math. Soc.*, vol. 55, 1949, pp. 797–800] with Laguerre polynomials.

6. *An approach to determinants via matrices*, by Professor Frank Levin, University of Kentucky, introduced by Professor A. W. Goodman.

The relationship between multiplication of determinants and of the matrices associated with them provides the well-known rules regarding row and column operations on determinants. This relationship can be made geometrically intuitive by a consideration of the effects of various linear transformations of the volume of a parallelepiped. Further, the linear transformations yield the rule for matrix multiplication. The paper gives an elementary method of proof of the various geometrical propositions involved.

7. *Recent trends in freshman mathematics programs*, by Professor W. L. Moore, University of Louisville.

The speaker reported on his visits to the University of Chicago and Case Institute of Technology during a semester's leave of absence. A symposium on the freshman program by Professor Price of the University of Kansas was held at the Institute. Two programs suggested to Dr. Moore are to be tried at the University of Louisville in the fall semester. One of these, suggested by the Chicago staff, will give an introduction to the principles of mathematics and the other a course in calculus not requiring analytic geometry as a prerequisite.

A. W. GOODMAN, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at Fontbonne College, St. Louis, Missouri, on April 21, 1956. Professor H. D. Brunk, Vice-Chairman of the Section, and Miss Frances Story, Chairman of the Missouri Council of Teachers of Mathematics, presided at two parallel morning sections. Professor Francis Regan, Chairman of the Section, presided at the afternoon session. There were 86 persons in attendance, including 42 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor R. J. Michel, Southeast Missouri State College; Vice-Chairman, Professor W. R. Utz, University of Missouri; Local Secretary-Treasurer, Professor C. H. Dalton, Southeast Missouri State College. Professor Margaret F. Willerding, Harris Teachers College, was re-elected Association Secretary for a period of another five years.

The following papers were presented:

1. *A report of a conference between high school and college teachers of mathe-*

matics and science, by Professor C. A. Johnson, University of Missouri, School of Mines and Metallurgy at Rolla.

A joint panel discussion by college professors and high school teachers concerned three problems: (1) How can we induce more students to complete high school? (2) How can we revise or revitalize the present high school curriculum to make it suitable to progress in scientific areas such as engineering? (3) How can we secure the personnel to carry out the teaching of science and mathematics on the high school and college level, especially in view of the salary inducements provided by industry?

2. *College arithmetic for prospective teachers*, by Professors Marie A. Moore and Jesse Osborn, Harris Teachers College, presented by Professor Moore.

The results of Arithmetic Proficiency Tests given at Harris Teachers College have proven that many of the students are poorly equipped in arithmetic content. The need of more effort to strengthen these students in the content before they attempt courses in methods was emphasized. Permanent progress in teaching methods must begin with and must not lose contact with the experiences and judgments of successful teachers of all times. Such progress has been hindered by promoters of catch-words urging swings from one extreme position to another.

3. *The Boolean geometry of the integers*, by Professor J. L. Zemmer, University of Missouri.

An abstract set S is called a Boolean space if there is a function $d(x, y)$ defined on the set of pairs of elements of S with values in a Boolean algebra B and satisfying the usual requirements of a distance function, namely, (1) $d(x, y) = d(y, x)$; (2) $d(x, y) = 0$ if and only if $x = y$; (3) $d(x, y) \subseteq d(x, z) \cup d(z, y)$. In this paper a distance function is defined on the ring of integers with values in a suitable Boolean algebra. Some of the geometric properties of this Boolean space are discussed. The main result is a description of the group of isometries.

4. *Analytical functionals and symbolic calculus*, by Professor Maria Castellani, University of Kansas City.

The speaker outlined the theory of Fantappie's indicatrices and their basic concepts and applications to the theory of analytical functionals and symbolic calculus. A special consideration was given to the use of these indicatrices in methods of solving partial differential equations.

5. *Recreational mathematics for use in the elementary classroom*, by Mrs. Ruth H. Nies, Ladue School System, introduced by the Secretary.

Recreational arithmetic makes use of entertaining materials which stimulate interest, create favorable attitudes, organize and review fundamental processes, and strengthen basic concepts. Classroom experiences have shown that recreational methods, properly used, result in improved achievement as evidenced on standardized tests and in practical application of problem solving. Students at all grade levels may benefit from a recreational approach, which is worthwhile for the average and remedial groups as well as for those in an enrichment program. The values of recreational arithmetic in modern teaching should not be overlooked.

6. *What position for recreation in high school classrooms?* by Mrs. Mattie B. Ryland, Higginsville High School, introduced by the Secretary.

Adaptation of teaching methods to meet needs of the high school mathematics student may well include increase in the use of and variety of recreational mathematics. Material used must prove enjoyable to the student lest he consider the term a misnomer. Use provokes questions. Shall all be required to participate? Shall the purpose be motivation, a method used with expected regularity, a guard against loss of interest, or an insert between units? Careful selection of material

and consideration of these questions by the instructor will determine position in the individual classroom.

7. *Glimpses of mathematical recreations on the college campus*, by Professor Margaret F. Willerding, Harris Teachers College.

The speaker discussed two phases of recreational mathematics on the college campus. The first phase was an outline of an in-service course for teachers called "Recreational Mathematics for Use in the Elementary and Secondary Classroom". The second phase was the use of recreational mathematics as a motivating factor in undergraduate courses of mathematics. Recreational mathematics was particularly stressed in teacher training courses.

8. *Mathematics in the liberal arts curriculum*, by Professor A. E. Ross, Notre Dame University. (By invitation.)

In the present lecture we propose what we feel to be a well rounded and effective program of training in mathematics for non-mathematical specialists. The proposed program begins with the study of the role of language as a means of communicating ideas, of the role of technical terminology and of the very basic type of mathematical experience leading to the study of the algebra of classes and the Boolean algebra. The program continues with the study of number theory, projective geometry, real numbers, vectors, analytic geometry, and complex numbers.

Some basic ideas of calculus are introduced at the conclusion of the course. Without minimizing the importance of the acquisition of manipulative skills of mathematics, one must realize that mathematics achieves the standing of a liberal arts subject only if the teaching reaches beyond manipulation into the wealth of mathematical ideas and if it does not neglect to bring out the dynamic or creative side of the subject.

MARGARET F. WILLERDING, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-second annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on April 21, 1956. Professor L. K. Jackson of the University of Nebraska presided.

Thirty-eight persons attended the meeting. Of these, twenty-two were members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor J. M. Earl, University of Omaha; Vice-Chairman, Professor L. K. Jackson, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *Infinite exponentials*, by Mr. D. W. Miller, University of Nebraska.

By analogy with infinite series and infinite products, we define infinite exponentials in the following manner. Suppose given a countably infinite sequence $a_1, a_2, a_3, \cdots$ of non-negative real numbers. For real numbers a and b, with a, b not both zero, we denote a^b by $[a, b]$. Now define the sequence $X_1, X_2, X_3, \cdots$ by

$$X_n = [a_1, [a_2, [a_3, \cdots, [a_{n-1}, a_n]] \cdots], \qquad n = 1, 2, 3, \cdots.$$

Then $[a_1, [a_2, [a_3, [\cdots$ is called an infinite exponential written $X_{i=1}^{\infty} a_i$. If $\lim X_n = E < \infty$, we say that the exponential converges to E, otherwise we say that the exponential diverges. Certain special classes of infinite exponentials are considered, and questions of convergence and divergence are discussed.

2. *About the arithmetic and geometric mean*, by Professor Arne Magnus, University of Nebraska, introduced by the Secretary.

Various arithmetic and geometric proofs of the inequality between the arithmetic and geometric means and related inequalities were presented.

3. *A locus on the sphere*, by Professor O. C. Collins, University of Nebraska, introduced by the Secretary.

In a circle, a chord TPT', perpendicular to the radius through O, subtends a central angle $2K$. A sphere touches the plane of the circle at O. S and S' are those points on the sphere which are nearest to T and T'. In the plane of the circle lies an ellipse, which has for its major axis that diameter of the circle which passes through O, and for the value of its eccentricity the sine of one half of the arc SS' of the sphere. A point P, on the sphere, is determined so that the foot of a perpendicular let fall from it onto the plane of the circle lies on the ellipse. It is proved that in the spherical triangle SPS' the angle at P is equal to K.

4. *New mathematics courses for freshmen and sophomores*, by Professor G. B. Price, University of Kansas. (By invitation).

5. *Lattice of divisibility and multiplication of the natural numbers*, by Professor H. B. Ribeiro, University of Nebraska.

6. *Functions defined by integrals*, by Professor W. G. Leavitt, University of Nebraska.

The method of defining a function by means of an integral is illustrated in a simple way by considering such a definition for $\log x$. Many of the properties of the logarithm can then be derived by using elementary properties of integrals.

7. *Quasi-convex subsets of finite projective geometries*, by Dr. E. J. Schweppe, University of Nebraska.

A set Q of points in a projective space is called quasi-convex if at least one of the points C, D is in Q whenever C and D are separated harmonically by points A and B in Q. It is easily shown that the pointwise complement of a quasi-convex set is also quasi-convex. Using this result it is shown that, in a projective geometry which contains only a finite number of points, all quasi-convex subsets are almost trivial and those which do exist are then described explicitly.

8. *The decline and fall of the general solution empire*, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

Since only a few of the simplest problems of Diophantine analysis can have a general solution in polynomials, a more realistic program must be set up. A problem is solved through n if all solutions $T(n)$ in integers equal to or less than n are tabulated. Let the number of solutions covered by a particular algebraic formula, and all such formulas known, be $A'(n)$ and $A(n)$ respectively. Then (adapting an idea from Dickson's *History* II, xx) the virility of the particular formula is $V'(n) = A'(n)/T(n)$, or $V(n) = A(n)/T(n)$. In most problems of plane and solid arithmogeometry, the virility of algebra is a small percent.

EDWIN HALFAR, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-fifth annual meeting of the Illinois Section of the Mathematical Association of America was held at Eastern Illinois State College, Charleston, Illinois, on May 11 and 12, 1956. Dean Hugh Beveridge, Chairman of the Sec-

tion, presided at all sessions.

There were 63 persons in attendance, including 43 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor F. E. Hohn, University of Illinois; Vice-Chairman, Professor C. T. McCormick, Illinois State Normal University; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. Professor Arnold Wendt reported on the work of the Committee on Contests and Awards stating that about 1000 more high school students participated this year than a year ago. Professor L. A. Ringenberg reported that the Committee on the Strengthening of the Teaching of Mathematics in conjunction with a committee representing the Illinois Council of Teachers of Mathematics were working cooperatively with the State Office of Public Instruction in arranging for correspondence work in higher mathematics in the smaller schools. Among other things, they were making it easier to publish articles on the teaching of mathematics in elementary and secondary schools.

The following program was presented:

1. *Three problems*, by Professor A. W. McGaughey, Bradley University.

This paper presented three problems leading to the same two pairs of algebraic equations in two unknowns. Four pairs of solutions were obtained, one pair being positive integers, another pair being rational but non-integers and the other two pairs being complex numbers. It pointed out that the positive integers were the only solution for one problem, the rational pair could be used also for the second problem. It then showed how the complex numbers could be interpreted as real solutions for the third problem.

2. *Preface to undergraduate research*, by Miss Rose Lariviere, University of Illinois, Navy Pier.

Some important features of the training required by the immature research worker were discussed, and group projects for elementary college classes, intended to increase confidence and to improve mathematical style, were described.

3. *Differentials*, by Professor M. E. Munroe, University of Illinois, introduced by the Secretary.

A valuable part of calculus, the algebra of differentials, is usually presented as suggestive only and scrupulously avoided in rigorous treatments. Development of calculus is strengthened by introducing the theory of differentiable manifolds in a suitably specialized, concrete form. Even this theory is enhanced by using the differential in the definition of the integral. Replace the usual differences $x_i - x_{i-1}$ by values of the operator dx on segments of the tangent line. When this is properly done, a rigorous proof of the substitution formula becomes trivial. A similar treatment of multiple integrals makes obvious the Jacobian formula for change of variable.

4. *Scientists and the secondary schools*, by Dr. J. R. Mayor, Director, Science Teaching Improvement Program, American Association for Advancement of Science.

Professional scientific organizations are now more deeply concerned about science and mathematics education in the secondary schools than at any other time in this century. Scientists recognize that they have a responsibility for the improvement of secondary-school teaching which has

not been adequately met. Throughout the nation there is evidence of new interest in teacher education and of a desire on the part of college and university staff members in academic departments to cooperate with secondary-school teachers and administrators in improving secondary-school programs. If this interest and spirit of cooperation can be properly coordinated and fostered, desirable modification of some goals and trends in secondary education that have been questioned by scientists could result.

5. *Gunderson-Hoffman semi-groups*, by Mr. Gayne Gunderson and Mr. D. K. Hoffman, presented by Professor Gertrude Hendrix, Eastern Illinois State College.

Gunderson and Hoffman have found that the "set of eight numbers" reported by Porges (this MONTHLY, vol. 52, 1945, pp. 379–382) exists not because of peculiar properties of those eight natural numbers, but because of the base-ten notation in which they are symbolized. Using Porges' operator on numbers written to other bases reveals a large family of semi-groups, probably infinite. In their paper, *Groupoids determined by summing squares of digits of a natural number written to an arbitrary base*, theorems are proved establishing that exhaustive results for each base can be obtained by considering only numbers less than the square of the base. Two theorems predicting elements of two semi-groups for each odd base also are proved.

Further investigation takes two directions: (1) search for further relations by which elements of semi-groups for a given base can be predicted; (2) consideration of semi-groups which may appear when higher powers of the digits are summed.

6. *Mathematics clubs in Illinois*, by Professor D. R. Bey, Illinois State Normal University.

The present status of mathematics clubs is important because it reflects educational philosophy. At the time of the state-wide study during the school year 1954–1955 about one secondary school in ten had an active club in mathematics; indications were that the number of mathematics clubs was decreasing. In schools where clubs were in existence, teachers placed a higher evaluation on their contributions to the mathematics program than did the teachers in schools without clubs. There is a need for a state association of mathematics clubs in order to establish a unified policy, strengthen the programs of individual clubs, and encourage the formation of new clubs.

7. *A mathematical introduction to logic circuits*, by Professor F. E. Hohn, University of Illinois.

This paper outlines those aspects of propositional logic which can be represented by electronic circuits and shows how logical diagrams for these circuits may be developed.

A. W. MCGAUGHEY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section was held in conjunction with the annual meeting of the Minnesota Academy of Science at Augsburg College in Minneapolis on May 5, 1956. There were 64 persons attending the meeting including 54 members of the Association.

Professor George Soberg of Augsburg College presided at the morning session and Professor E. J. Camp of Macalester College, the retiring Chairman of the Section, presided at the afternoon session.

At the business meeting, the following officers were elected to serve during the coming year: Chairman: Professor Walter Fleming, Mankato State Teachers College; Secretary-Treasurer: Professor F. C. Smith, College of St. Thomas;

Executive Committee: Professor E. J. Camp of Macalester College, Professor W. B. Fulks, University of Minnesota, and Professor F. L. Wolf, Carleton College.

By invitation of the Executive Committee, Professor W. L. Hart of the University of Minnesota delivered an address entitled "Certain Iterative Gradient Methods for the Solution of Systems of Equations." The abstract of this address follows:

With $x=(x_1, x_2, \cdots, x_n)$ in real n-space, $f(x)$ real valued, and $x^{(0)}$ a given point, the author first described the well-known vector correction $\Delta x^{(0)}$ in the path of steepest descent to obtain $x^{(1)}=x^{(0)}+\Delta x^{(0)}$ as a next approximation to a solution of $f(x)=0$. Then, the paper discussed two methods for obtaining a sequence $\{x^{(m)}\}$ converging to a solution of a system $f(x)=0$, $f=(f_1, \cdots, f_k)$, where gradient corrections are involved, as follows. (a) The sequential projection method (due to S. Kaczmarz for linear systems and to C. B. Tompkins for general systems). (b) The composite Newton-Raphson gradient method (due to W. L. Hart and T. S. Motzkin) which reduces, for linear systems, to a special case of a method for the linear case due to M. R. Hestenes and M. L. Stein. The method of proof for convergence of $\{x^{(m)}\}$ in (b) was outlined.

The following short papers were presented:

1. *A measurable mapping from one circle into another*, by Professor Hidehiko Yamabe, University of Minnesota.

Let C denote the circle group, *i.e.*, the real numbers mod 1. We can define a mapping of C into itself associated with an integer m which maps x into mx. Then, for any two given circles, say C_1 and C_2, does there exist a measurable mapping f so that $f(mx)=n(fx)$ holds for almost all x on C_1? For the case $m \geqq n$, there exists such a mapping f with meas $(f(x), x \in C_1)=1$. The author proposes a conjecture that when $m<n$, meas $(f(x), x \in C_1)=0$ if such a mapping f exists.

2. *An application of mapping*, by Mrs. Hildegarde Howden, St. Olaf College.

Proposed: To show that the laws of the 12-tone technique of musical composition form a group by satisfying the following conditions: (1) The identity case 0 belongs to the set T of transformations; (2) If R (In, RIn) belongs to T, then its inverse $R^{-1}(In^{-1}, (RIn)^{-1})$ does; (3) If R and In (or R and RIn, In and RIn) belong to T, then the composite RIn does. In general, composition of mappings is not commutative; however, the transformations in the 12-tone technique of musical composition are commutative, *i.e.*, $RIn=InR$.

3. *Some elementary aspects of linear programming*, by Mr. J. S. Hill, Actuary, Minnesota Mutual Life Insurance Company, St. Paul, Minnesota.

The most popular text on the subject is by Charnes, Cooper and Henderson, and it was prepared from materials developed in a research seminar at the Carnegie Institute of Technology. The "simplex method" of computation is credited to G. B. Dantzig of the Rand Corporation. The subject has sparked the interest of economists, engineers, statisticians, mathematicians, and persons associated with business and industry.

The theory rests on the proposition that within an n-dimensional space there can be constructed a convex polyhedral cone whose boundaries are determined by the limitations which are basic to the problem. It is then possible to proceed from point to point in a manner such that the distance from the origin (*i.e.*, the value of a linear functional) is never decreased. Thus by a methodical process, a maximum value of the functional is obtained. The process consists of setting up a matrix or "simplex tableau" and using specific rules and formulae to transform the matrix by successive steps. At each step a simple test is applied to see if the maximum has been reached.

The types of problems to which the technique has been applied are machine productivities and capacities, optimum storage, shipment and distribution of goods, minimizing set-up times in

machine shop operations, and optimal labor allocation. Military combat and logistics problems, personnel selection, procurement, transportation, and Leontif-type problems are also actively being explored with this technique. The author hopes to experiment in certain life insurance management problems. As in all areas of applied mathematics, the highest order of skill is required in setting up the problem, developing reliable functions for the variables and in making a valid interpretation of the result.

4. *A panel discussion on the state committee for evaluation of secondary mathematics*, by Professor D. A. Johnson, University of Minnesota, Mr. Clarence Olander, St. Louis Park High School, and Professor F. L. Wolf, Carleton College.

The panel discussed the efforts of a state committee to develop methods and procedures for studying and improving the mathematics program in the secondary schools of Minnesota. The panel presented the suggested methods which the committee has developed to date and asked the audience for comments and criticisms to guide the committee in its future work.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA, Concordia College, Moorhead, October 6, 1956.

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA, Oklahoma City University, October 26, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pa., November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

HOMOLOGICAL ALGEBRA

By Henri Cartan & Samuel Eilenberg

During the last decade the methods of algebraic topology have invaded the domain of pure algebra and initiated a number of internal revolutions. This book presents a unified account of these developments and lays the foundation of a full-fledged theory.

Princeton Mathematical Series, #19. 416 pages. $7.50

INTRODUCTION TO MATHEMATICAL LOGIC (Vol. I)

By Alonzo Church

The long-awaited basic treatise on mathematical logic, which begins on an elementary level suitable for students and progresses rapidly to more advanced results. *Princeton Mathematical Series, #17. 400 pages. $7.50*

Order from

PRINCETON UNIVERSITY PRESS
Princeton, New Jersey

NEW REPRINT

Now Available

American Mathematical Monthly

Volumes 1-20, 1894-1913

Cloth bound set	$275.00	
Paper bound set	245.00	
Volumes 1-9, paper bound		$15.00 each
Volumes 10-20, paper bound		10.00 each

This reprint is being undertaken with the permission of the Mathematical Association of America. Please address orders and inquiries to

JOHNSON REPRINT CORPORATION
125 East 23 Street
New York 10, New York

APPLIED MATHEMATICIAN
to $11,500

in this stimulating
Missile Test Project

You will be challenged by the research and theoretical studies involved in acquiring data from high velocity missiles being fired over the world's longest test range. Ph.D. degree plus several years' experience in work related to above, required.

Join a scientific team at top level in this unprecedented work with one of the nation's largest corporations. Ideal living and working conditions on Florida's Central East Coast.

TO ARRANGE CONFIDENTIAL INTERVIEW

Send resume to Mr. H. C. Laur, Dept. N-11H
Missile Test Project
PO Box 1226
Melbourne, Florida

APPLIED MATHEMATICIANS

The Jet Propulsion Laboratory is working on many challenging problems relating to all phases of jet propulsion, aerodynamics and missile control. This work is supported by modern computer facilities together with excellent staff accommodations.

The Laboratory is a continuing operation devoted to scientific research and development offering many opportunities for increasing responsibilities with its expanding activity.

In our Machine Computing Group we now have several such opportunities open for Master's-degree applied mathematicians for the mathematical analysis of advanced engineering problems. Machine computing experience is helpful but not essential.

If you are interested in such work, in pleasant surroundings, please send us your qualifications immediately.

jpl

JET PROPULSION LABORATORY

A DIVISION OF CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

The First in a Series of Announcements on Progressive Expansion of Program and Facilities in Mathematics at the Knolls Atomic Power Laboratory:

GENERAL ELECTRIC'S
KNOLLS ATOMIC POWER LABORATORY

Announces

CONSTRUCTION OF A MODERN CENTER FOR MATHEMATICS

Because we believe that theory is our most powerful weapon in dealing with reality, we are expanding our Mathematical Analysis Program. One of the first elements in this expansion is the creation of a new and modern building for mathematicians and physicists, which will be the center of the Laboratory's efforts to meet by theoretical means the challenges of the nuclear energy field.

We are seeking men with strong mathematical training at all degree levels to participate in this expanding Numerical Analysis Program—a program growing not only in staff, equipment, and facilities, but also in concept and function. They will work in close association with our theoretical and experimental physicists. There are openings in each of the following fields:

RESEARCH IN MATHEMATICAL TECHNIQUES

Numerical solution of the diffusion equation for complicated geometrical arrays taxes even the most powerful electronic computers. Fundamental work in iterative techniques must be carried out.

FORMULATION AND EVALUATION OF THEORIES

Due to the nature of physical situations now being encountered, the rough approximations which were formerly adequate must now be improved. The ultimate test of such improvement is comparison with experiment.

APPLICATIONS TO REACTOR PROBLEMS

A broad program of computational tools for reactor design must be effected incorporating the best available techniques. Strong interests in computation and in machine properties are indicated. The program at Knolls offers the atmosphere, the equipment, the richness of subject matter and the material benefits conducive to a satisfying career in applications of mathematics.

A LETTER TO DR. S. R. ACKER, EXPRESSING YOUR INTEREST, WILL RECEIVE IMMEDIATE ATTENTION.

Knolls Atomic Power Laboratory

GENERAL ELECTRIC

SCHENECTADY, N. Y.

MATHEMATICIANS

For Analysis Group of expanding research and development laboratory. Principal fields of interest are: weapons systems analysis, peacetime applications of atomic energy, and operations research. Several openings are available.

1 To carry out studies in OPERATIONS RESEARCH. Familiarity with probability theory, linear programming, game theory, information theory, optimization procedures, and other OR techniques very desirable.

2 To perform operational analyses requiring extensive background in aerodynamics and exterior ballistics. May also investigate problems dealing with missile fire control and navigational systems. Familiarity with digital computer programming desirable.

3 To conduct investigations in the fields of electromagnetic theory, acoustics, thermal and radiation effects.

These openings require men with vision and initiative. Our modern laboratory provides a professional working atmosphere and the location in a quiet suburban area provides pleasant living and working with easy access to the cultural and educational facilities of New York City.

All inquiries in confidence. Please send resume, including salary desired, to Personnel Manager.

VITRO LABORATORIES

Division of Vitro Corp. of America

200 Pleasant Valley Way
West Orange, New Jersey

Engineers • Mathematicians • Physicists

TRANSLATE ADVANCED ENGINEERING INTO MATHEMATICAL TERMS

Professional engineers and scientists with a strong interest in digital applications are needed at GE's expanding Aircraft Nuclear Propulsion Department. Advanced engineering or scientific knowledge is necessary, as well as the ability to analyze engineering problems. While computer experience is desirable, it is not essential.

Specifically, the man who undertake this work must be able to take a physical problem and translate it into mathematical terms suitable for digital programming analysis and solution.

Some of the most extensive and modern computer facilities in industry are at your service at GE's Cincinnati plant. You may also do graduate work in your field under GE's re-imbursed tuition plan. And the city itself, one of the country's well-known cultural centers, is a fine place to live.

Please write full details in confidence, including salary requirements to:

MR. J. R. ROSSELOT
Aircraft Nuclear Propulsion Dept.

GENERAL ELECTRIC

Cincinnati 15, Ohio

OUTSTANDING MATHEMATICS TEXTS

Analytic Geometry and Calculus

W. R. Longley, P. F. Smith, W. A. Wilson

Combines two subjects to meet needs of science and engineering students, with integration introduced early. Many worked-out examples; chapter on differential equations.

Elements of Calculus

W. A. Granville, P. F. Smith, W. R. Longley

Gives applications of differential and integral calculus early, after developing formulas, to aid physics and engineering students. Graded problems; clear presentation.

Elements of the Differential and Integral Calculus, Revised Edition—*Granville, Smith, Longley*

This well-known, standard text teaches differentiation and integration separately.

Advanced Calculus—*A. E. Taylor*

Stresses understanding of concepts and basic principles of analysis—properties of the real number system which support the theory of limits and continuity.

Please Ask for Descriptive Circulars

GINN AND COMPANY

Sales Offices:
New York 11
Chicago 6
Atlanta 3
Dallas 1
Columbus 16
San Francisco 3
Toronto 7

Home Office:
Boston

Worth Your Attention

The new ENGINEERING MATHEMATICS *by Kenneth S. Miller* NEW YORK UNIVERSITY

This modern, thorough treatment of differential equations, network theory and random functions provides the mathematics needed for today's engineering research and development projects. Much material not heretofore found in texts on this subject is included, as, for instance, the Frobenius method, the Hermite, Laguerre and Tchebycheff functions, Green's function, and Weiner-Khintchine relations.

Copies will be available in October. Is your name on our waiting list to receive a text examination copy on publication?

The new CALCULUS *by Jack R. Britton*

UNIVERSITY OF COLORADO

Based on years of careful planning, class-testing and teaching experience, and written by an author whose other mathematics texts are known for their clarity and thorough teaching of fundamentals, this new unified treatment of differential and integral calculus is outstandingly well organized, easy-to-understand, and at the same time mathematically rigorous. Contains hundreds of well-graded exercises, including interesting problems of a theoretical nature and review questions designed to develop the student's critical and analytical faculties.

Over 1200 teachers have already requested copies for text consideration. If you are not among them, send now for your examination copy of this outstanding new CALCULUS.

Did you know—

that FUNDAMENTALS OF COLLEGE MATHEMATICS by Johnson, McCoy & O'Neill

—one of the first and best of the unified, rigorous presentations of basic mathematical concepts—is now available at $5.00?

RINEHART
& COMPANY, INC.
232 Madison Ave., New York 16

6

MATHEMATICS TEXTS BY

PAUL R. RIDER

Professor Emeritus, Washington University, and Statistical Advisor, Wright-Patterson Air Force Base, Dayton, Ohio

COLLEGE ALGEBRA, Revised Edition

Retaining the first edition's clear style, this text features an expanded discussion of basic algebraic concepts, completely new sets of exercises, and an improved method of presenting generalized exponents.

1955 352 pp. $4.00
Alternate Edition: 1947 407 pp. $4.00

PLANE TRIGONOMETRY

This text presents trigonometric functions in terms of their right-triangle definitions before the general trigonometric formulas and identities. Answers to odd-numbered exercises are included. *1953 180 pp. $3.00*

PLANE AND SPHERICAL TRIGONOMETRY

Here is a complete text, adaptable to any order of presentation. The first problems are simplified, numerically, so that the student can grasp principles and learn methods. Formulas are developed as needed.

1942 With tables 418 pp. $3.90
Without tables 275 pp. $3.40

FIRST YEAR MATHEMATICS FOR COLLEGES

This single volume contains all the basic topics for the first year course in mathematics. Each chapter is sufficiently developed so that the teacher can adapt the book to his own sequence. *1949 714 pp. $5.00*

INTERMEDIATE ALGEBRA FOR COLLEGES

Intended for students having only one year of high school algebra, this book is noted for clear exposition and an abundance of carefully graded, accurate problems. *1949 242 pp. $3.50*

ANALYTIC GEOMETRY

Written in easily understood language, this book makes the student aware of the practical applications of the material he is studying, with no sacrifice of sound mathematical rigor. Answers to odd-numbered exercises are given in the book. *1947 383 pp. $4.00*

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N.Y.

Distributors in Canada: BRETT-MACMILLAN LTD., 25 Hollinger Road, Toronto

Outstanding McGRAW-HILL Books

FUNDAMENTAL MATHEMATICS

By THOMAS L. WADE and HOWARD E. TAYLOR, The Florida State University. 374 pages, $4.75

Readily adapted to any classroom need though primarily a basic text for students with inadequate secondary school preparation in mathematics. Also provides ample foundation material for study in the social sciences, physical sciences, education, and business. Fundamental ideas of elementary algebra are developed in a logical and orderly manner . . . each operation treated first for the numbers of arithmetic, then for the literal symbols of algebra . . . designed to improve understanding of the basic principles underlying the fundamental operations with numbers.

ENGINEERING ANALYSIS: A Survey of Numerical Procedures

By STEPHEN H. CRANDALL, Massachusetts Institute of Technology. 430 pages, $9.50

A second-year graduate text devoted to the advanced problems of engineering analysis. Considers briefly the formulation of mathematical models and the construction of computational programs for solving these with automatic or hand computing machines. The material is arranged according to a natural classification of the basic types of problems in engineering analysis, including equilibrium, eigenvalue, and propagation. It is the first book from an engineering viewpoint at this advanced level.

PHYSICS AND MATHEMATICS Progress In Nuclear Energy—Series I

Edited by R. A. CHARPIE, Oak Ridge; D. J. HUGHES, Brookhaven; D. J. LITTLER, Harwell; and S. HOROWITZ, Saclay. 408 pages, $12.00

This volume presents a summary of results, and methods for their analysis, pertaining to the properties of the fissionable nuclei. The latter part of the volume includes material on Reactor Physics, and involves the techniques used in planning experiments with reactors. The results and methods presented will be of great value to workers in reactor design. Here is a careful evaluation and distillation of material by experts in the field.

STATISTICS FOR ECONOMICS AND BUSINESS . . .
New Second Edition

By DONALD W. PADEN, University of Illinois, and E. F. LINDQUIST, University of Iowa. 320 pages, $4.75

A concise, non-technical interpretation of the common statistical tools currently used in business and economics. Computation and mathematical manipulation are held to a minimum. Sampling, correlation, and time series are covered with more than the usual care. The *Student Workbook*, correlated with the text, contains problems to help the student formulate the principles to be learned. The package provides a basic understanding of the more important statistical techniques required in the business world.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN